Weighted Consecutive-Loss Rules For Justified Representation in Perpetual Voting

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Abstract

Existing discussions of voting rules and group fairness in approval voting have been largely limited to single-stage settings where elections are considered in isolation. We extend this discussion by building upon the incipient literature on perpetual voting, which considers voting mechanisms and fairness guarantees in multi-period settings. We introduce criteria to characterize weighted voting rules that are intuitively appealing and tractable in the perpetual setting. We then investigate the properties of such rules in relation to justified representation (JR) and extended justified representation (EJR). In particular, we formulate classes of consecutive-loss based weighted voting rules, and identify constraints under which such rules satisfy or approximate JR and EJR. Finally, we test voting rules which empirically satisfy or approximate properties of fairness with perpetual voting simulations.

1 Introduction

While fairness properties in single-stage elections are well-studied, our conception of fairness might extend beyond one period. Consider a sequence of several elections between two candidates with the winner determined by majority. One candidate might occupy a 49% minority in each, yet lose all the elections. A mechanism that seems fair when each election is considered individually may be unfair to minorities when applied over a sequence of elections.

Perpetual voting rules, proposed by Lackner [2020] and Lackner and Maly [2021; 2023], are a class of voting rules that operate over multi-period settings, considering the history of voters' electoral representation as inputs to the outcome of the current election. These rules offer a mechanism to satisfy intertemporal conceptions of fairness by providing some representation for minorities across periods, since voters who have been left unsatisfied in previous elections may be weighted more highly than those who have been satisfied.

As with Lackner and Maly [2023], we start with the important desideratum of perpetual voting rules that they should be simple and interpretable. Considering what Lackner and Maly [2021] call "basic weighted approval methods" that simply reweight voters after each election, we propose that we can distinguish the complexity of these basic rules by the growth rate in equivalence classes of the reweighting function. We focus on a class of particularly simple basic rules by this metric—rules which determine weights based on voters' number of consecutive losses—and investigate the extent to which they achieve notions of proportional representation in the multi-period setting.

This paper advances the literature on perpetual voting by filling in two gaps in prior work. First, to our knowledge, this paper is the first rigorous treatment of justified representation (JR) and extended justified representation (EJR) in the online multi-period setting with dynamic preferences. Since none of the rules suggested by Lackner and Maly [2021; 2023] achieve EJR, they contend that perpetual rules should be evaluated in relation to weaker criteria like apportionment lower quota. However, since JR and EJR are well-established in the single-period problem of selecting a k-member committee, it is natural to study whether similar group fairness guarantees can hold when preferences change over multiple elections. Bulteau et al [2021] suggest multi-period versions of JR and EJR, but they consider the easier setting of offline preferences, deviating from the intention of Lackner and Maly's original perpetual voting framework. We believe our work on JR and EJR in online perpetual voting opens up this direction for further research.

Second, we make progress on the front of discovering interpretable perpetual rules by demonstrating that consecutive-loss-based basic rules approximate proportionality. Lackner and Maly [2023] show negative results for win-based and loss-based basic rules, and take these results to motivate the study of non-basic rules, such as Phragmen's rule, which are significantly more complex. However, there is much unexplored middle ground between the two extremes, and the Pareto frontier for the tradeoff between fairness and interpretability is unknown. Our proposed complexity metric gives us a tool for understanding this tradeoff, and our finding that consecutive-loss rules, which are the same level of complexity as Lackner's win-based and loss-based rules, can satisfy JR and approximate EJR, is a step toward developing perpetual rules that are both fair and interpretable.

1.1 Related work

Prior to Lackner [2020]'s work formalizing perpetual voting, several alternate approaches to the multi-period representation problem had been suggested, including Frege's method (Harrenstein [2020]), sequential Nash welfare maximization (Freeman et al [2017]), dynamic voting (Parkes and Procaccia [2013]), and storable votes (Casella [2012]). These frameworks were created to study objectives other than voter group fairness and/or require substantially more information about preferences than perpetual voting.

Since Lackner [2020]'s introduction of perpetual voting, more work has studied online-preference voting models, particularly online approval-based committees (Do et al [2022]; Israel and Brill [2021]; Brill et al [2022b]; Imber et al [2022]). While these papers achieve positive results for fairness guarantees, their models differ substantially from the perpetual voting setting; the target committee size is known and preferences are revealed for one candidate at a time, so candidates are elected to the committee in only a subset of the periods.

2 Preliminaries

2.1 Perpetual voting

Following Lackner and Maly [2021; 2023], we consider an online model with dynamic preferences in which future candidates and preferences are not known and the winner in each period is determined solely based on previous and current preferences and previous winners. Denote $\mathbb{N} = \{1, ...\}$, $[k] = \{1, ..., k\}$, $a_{1:k} = (a_1, ..., a_k)$, and let $\mathcal{P}(S)$ be the power set of S. In each period i, a set of voters V submits approval ballots for a set of candidates C_i. Though the voter set is fixed between periods, the candidate set need not be. Let $A_i : V \to \mathcal{P}(C_i)$ denote the approval ballots in period i.[1] The tuple (V, A_i, C_i) defines a *decision instance* for period i, which can be thought of as a single election in a series. A k-period sequence of elections and voting preferences can be written as a *k-decision sequence* $D = (V, \overline{A}, \overline{C})$, containing a set of voters V, a k-tuple of candidate sets $\overline{C} = C_{1:k}$, and a k-tuple of approval profiles $\overline{A} = A_{1:k}$, reflecting voter preferences over all k periods.

A perpetual voting rule is a specification for *all* $i \in \mathbb{N}$ of a map $(V, A_{1:i}, C_{1:i}, w_{1:i-1}) \mapsto w_i$. Iteratively applying a perpetual voting rule over a specific k-decision sequence produces a *winner sequence* $\overline{w} = (w_1, ..., w_k)$ of length k, but the rule itself must be defined to select candidates over *any* number of periods. Denote $W_i := \{v \in V \mid w_i \in A_i(v)\} \subset V$ the set of voters that approved the winner in period i.

2.2 Justified representation

We adapt the notions of JR and EJR to the perpetual setting. Since approval sets may shift, so may voter groups, creating ambiguity about which groups deserve representation. We borrow an "all-periods" interpretation from Bulteau et al [2021], under which groups need to be represented if their approval sets overlap in every period.

Definition 2.1 (Group cohesion)**.** A group of voters $V' \subset V$ is *cohesive* over k periods if $\forall i \in [k],\ \cap_{v \in V'} A_i(v) \neq \emptyset$.

Definition 2.2 (JR)**.** For a given k-decision sequence D, a *winner sequence* $\overline{w} = w_{1:k}$ satisfies Justified Representation (JR) if for every cohesive $V' \subset V$ such that $|V'| > |V|/k$, there exists some period j in which $w_j \in \bigcup_{v \in V'} A_j(v)$, i.e. some voter in the group is satisfied in at least one period.

Definition 2.3 (EJR)**.** For a given k-decision sequence D, a *winner sequence* $\overline{w} = w_{1:k}$ satisfies Extended Justified Representation (EJR) if for every cohesive $V' \subset V$ such that $|V'| > \ell \cdot |V|/k$, it holds that for some voter $v \in V'$, $|\{i \in [k] \mid w_i \in A_i(v)\}| \geq \ell$, i.e. some voter in the group is satisfied in at least ℓ periods.

Definition 2.4 (Voting rules)**.** A *voting rule* satisfies (E)JR if for every $k \in \mathbb{N}$, applying the rule to any k-decision sequence $(V, \overline{A}, \overline{C})$ produces a winner sequence that satisfies (E)JR.

The main difficulty of fairness in perpetual voting results from the combination of two circumstances. First, the candidate set and voter preferences may change between periods. Second, in addition to future candidates and preferences being hidden to the election mechanism, the time horizon (number of periods) is unknown in advance. Perpetual rules must be specified for all time horizons; our goal is to design mechanisms that achieve fairness guarantees considering the results of the elections that have occurred, regardless of when the election process stops. Rules must select winner sequences whose prefix sequences of any length satisfy (E)JR.

When either of these two degrees of freedom are restricted, rules that satisfy EJR can be constructed by extending insights from prior work on single-period committee elections. When candidates and preferences are static between periods, but the time horizon remains unknown, the model reduces to that considered by Brill et al. [2020; 2022a], which shows a committee-monotonic rule generalizing the D'Hondt apportionment method that achieves EJR. However, the behavior of this method is not well-defined when candidates and preferences change over time. On the other hand, if the time horizon of elections is known, a rule that accomodates dynamic preferences and satisfies EJR can be constructed by extending Peters and Skowron [2020]'s Rule of Equal Shares.

Definition 2.5 (Perpetual Equal Shares)**.** Each voter $v \in V$ starts with a budget $B_1(v)$ of 1. In each period i, the rule considers buying a candidate among the candidate set C_i. Buying a candidate costs $|V|/k$ dollars in total. If a candidate c is selected, the total cost is split among all voters approving c as evenly as possible – each voter either pays a universal amount q or the remainder of their budget $B_i(v)$, if $B_i(v) < q$. In each period, the rule selects the candidate among the C_i that minimizes q, and then budgets are adjusted accordingly. If no candidate is affordable even if all supporting voters pay their entire budget, the rule selects candidates arbitrarily.

Proposition 2.1. *Perpetual Equal Shares satisfies EJR.*

Both the rule itself and the proof that it satisfies EJR, provided in Appendix B, require minimal adjustment from Peters and Skowron [2020]'s original formulation, the main difference being that the candidate set and voter preferences change in every iteration. Notably, since Equal Shares is polynomial-time computable, Proposition 2.1 disproves Conjecture 1 in Bulteau et al [2021] that achieving PJR (a strictly weaker condition than EJR) when the time period is known is NP-hard.

[1]Unlike Lackner [2020], we allow voters to approve the empty set. Appendix A.1 explains why this does not affect any results.

However, the rule is immutably horizon-dependent, since the cost to buy any candidate $|V|/k$ reflects how much voters who have been satisfied must be penalized in favor of unrepresented voters – it is not true that winners in the first k periods will be the same if a longer time horizon is considered. For the rest of the paper, we return to the more general setting in which we assume rules are performed with respect to an unknown time horizon and seek to identify rules that satisfy fairness criteria regardless of the time horizon.

3 Basic rules

Our aim is to investigate the extent to which fairness guarantees can be achieved by simple rules that can be easily explained to voters. To begin, we revisit what Lackner and Maly [2021] call *basic* rules:

Definition 3.1. A *basic* perpetual rule satisfies the following:

- In period 1, each voter v_j has weight $\alpha_1(v_j) = 1$.
- In period i, each voter v_j has weight $\alpha_i(v_j) > 0$, there exist functions f_i, g_i such that $f_i(x) \geq x$ and $g_i(x) \leq x$ for all x, and [2]
$$\alpha_{i+1}(v_j) = \begin{cases} f_i(\alpha_i(v_j)), \ v_j \notin W_i \\ g_i(\alpha_i(v_j)), \ v_j \in W_i \end{cases}$$
For $G \subset V$, denote $\nu_i(G) := \sum_{v \in G} \alpha_i(v)$ to be the total weight of all voters in G in period i.
- A candidate wins if they receive the most weighted votes, with some method of breaking ties.

Basic rules are a particularly interpretable class of rules: they have sparse memory, i.e. they require minimal information about voter preferences in previous periods to compute (storing only a single numerical weight per voter); voters' weights are an intuitive proxy for how much their preferences matter in any election; and adding up votes is an especially canonical way to decide elections. Still, the space of basic rules is very large, and it is possible to select rules whose behavior is unintelligible.

We seek to develop a metric for a basic rule's level of complexity. Under Definition 3.1, two voters that have experienced the same pattern of "wins" (periods i in which $w_i \in A_i(v)$) and "losses" (periods i in which voter $w_i \notin A_i(v)$) up to but excluding period k must have the same weight in period k. On the other hand, the functions f_i, g_i may be specified so that any voters that have different win/loss records have different weights. Thus, a basic rule can equivalently be given the following structure:

Definition 3.2. Let a voter v's win/loss record before period k be encoded as a set $R_k := \{i \in [k-1] \mid v \in W_i\}$. A *basic rule* Φ is a specification for all $k \in \mathbb{N}$ of a map $\phi_k : \mathcal{P}([k-1]) \to \mathbb{R}^+$, such that $\phi_1(\emptyset) = 1$ and

- For $R \subset [k-1]$, $\phi_{k+1}(R) \geq \phi_k(R)$ and $\phi_{k+1}(R \cup \{k\}) \leq \phi_k(R)$.
- If $\phi_k(R) = \phi_k(R')$, then $\phi_{k+1}(R) = \phi_{k+1}(R')$ and $\phi_{k+1}(R \cup \{k\}) = \phi_{k+1}(R' \cup \{k\})$.

In period k, if voter v has win/loss record R_k, they have weight $\alpha_k(v) = \phi_k(R_k)$. A candidate wins if they receive the most weighted votes, with some method of breaking ties.

Under this formulation, it is clear that in period k, a basic rule partitions the voter set into up to $|\mathrm{Im}\phi_k| \leq 2^{k-1}$ equivalence classes of voters with the same weight, with each equivalence class in period k being a subset of some equivalence class in period $k-1$. It is thus possible to construct valid basic rules that utilize the entire permutation of a voter's win/loss record:

Example 3.1. *Let $\phi_k(R)$ be the numerical value of $R \setminus [k-1]$ treated as a binary representation, where period 1 is the least significant digit and period $k-1$ is the most significant.*[3]

On the other hand, the simplest rules throw away most of this information, and create far fewer equivalence classes.

Example 3.2. *Let $\phi_k(R) = k - |R|$.*[4]

We can therefore recognize the complexity of a basic perpetual rule by the growth rate of the number of equivalence classes the rule distinguishes as the number of electoral periods increases. Specifically, defining the *complexity function* c_Φ of a rule Φ to be the map $k \mapsto |\mathrm{Im}\phi_k|$ allows us to compare the complexity of two rules:

Definition 3.3. Given two basic rules Φ, Ψ, define Φ to be *more complex* than Ψ if $c_\Phi(k) \in O(c_\Psi(k))$.

Intuitively, a less complex rule uses less information about a voter's win/loss record to compute their electoral weight.

Finally, we restrict our consideration to rules for which voters' weights never increase if they counterfactually approve additional winning candidates, which are more well-behaved:

Definition 3.4. An *ordinal-preserving* rule has the additional constraint that if $R \subset R' \subset [k-1]$, then $\phi_k(R') \leq \phi_k(R)$.[5]

4 Linearly-bounded rules

According to the metric provided by Definition 3.3, the simplest rules to study are those in which the complexity function is linear ($c_\Phi(k) \in O(k)$). To construct such rules, we must classify win/loss records by some property that is supported on a set with cardinality linear in the number of periods. Four properties are naturally appealing: the total number of wins, the total number of losses, the number of consecutive wins since the last loss, and the number of consecutive losses since the last win. Lackner and Maly [2023] show negative results for win-based and loss-based rules with respect to some weak proportionality criteria. Indeed, we show that they cannot satisfy even JR without extreme distortion of voter weights.

Proposition 4.1. *Let Φ be an ordinal-preserving basic rule that satisfies JR for every way of breaking ties. For all $k \in \mathbb{N}$, the following condition must hold:*
$$\phi_k(\emptyset) > \sum_{i=1}^{k-1} \phi_k(\{i\})$$

[2]We allow f and g to depend on the period i, which is more general than Lackner and Maly [2021].

[3]The formulation of this rule under Definition 3.1 is as follows: let $f_k(x) = x + 2^{k-1}$; let $g_k(x) = x$.

[4]Under Definition 3.1: $f_k(x) = x + 1$; $g_k(x) = x$.

[5]Under Lackner and Maly's definition, rules for which f_i and g_i are always nondecreasing are ordinal-preserving.

The full proof in Appendix B involves considering a k-decision sequence in which k voters have disjoint preferences in the first $k-1$ periods, and the winner rotates through the preferred choice of the first $k-1$ voters; for JR to be satisfied, the winner in the kth period must be the kth voter's preferred choice even if all other voters rally around another candidate.

Corollary 4.1. *Let Φ be an ordinal-preserving basic rule that satisfies JR for some way of breaking ties. For all $k \in \mathbb{N}$, the following condition must hold:*

$$\phi_k(\emptyset) \geq \sum_{i=1}^{k-1} \phi_k(\{i\})$$

Though basic rules are generally sequences of maps $\phi_k : \mathcal{P}([k-1]) \to \mathbb{R}^+$, for rules with linear complexity it is often simpler to factor ϕ_k as a composition of maps $\tilde{\phi} \circ r_k$, where r_k is a "reduction" map taking win/loss records to a set with support with cardinality linear in k (for example, the map from a record to its total number of wins) and $\tilde{\phi}$ does not depend on k. The following are consequences of Corollary 4.1:

Proposition 4.2. *Let r_k map to the number of wins. No such rule satisfies JR, unless $\tilde{\phi}(k) = 0$ for all $k > 0$.*

Proposition 4.3. *Let r_k map to the number of losses. No such rule satisfies JR, unless $\tilde{\phi}(k) \geq k!$ for all k.*

Proposition 4.4. *Let r_k map to the number of consecutive wins. No such rule satisfies JR.*

Proposition 4.5. *Let r_k map to the number of wins, losses, or consecutive wins. No such rule satisfies EJR.*

These strong negative results motivate our study of consecutive-loss based rules. For the rest of the paper, we let r_k map to the number of consecutive losses.

5 Consecutive-loss rules

Our general strategy for proving results related to the JR and EJR properties of consecutive-loss based rules is to study an adversarial agent that tries to construct a decision sequence for which the winner sequence produced by the rule fails the relevant criteria. We suppose that a cohesive group $G \subset V$ large enough to deserve representation exists, but the adversary is allowed to select the preferences of the non-group voters in every period, and it attempts to manipulate the non-group voters' preferences in a way such that G is not sufficiently represented by the rule's chosen winner sequence. This setup enables us to treat JR and EJR-related proofs as constrained optimization problems.

5.1 Adversarial strategies

Consider JR satisfaction over k periods. Say $G \subset V$ is "represented" in period i if some $v \in G$ is satisfied (i.e. $w_i \in A_i(v)$).

Definition 5.1. For a k-decision sequence $D = (V_D, \overline{A}, \overline{C})$, a rule Φ, and a group $G \subset V_D$, an *adversarial strategy* $\mathcal{A} = (A_1|_{V\setminus G}, ..., A_{k-1}|_{V\setminus G})$ is a tuple of non-group approval profiles in periods $1, ..., k-1$ such that replacing non-group approval profiles in D with $\mathcal{A}$ and iterating Φ produces a winner sequence such that G is not represented in these periods. Abbreviate $\mathcal{S}_{D,G}$ the set of all adversarial strategies for (D, Φ, G) (where Φ is implicit).

Lemma 5.1. *A rule Φ satisfies JR iff for all $k \in \mathbb{N}$ and all k-decision sequences D and cohesive groups $G \subset V_D$ with $|G| \geq |V_D|/k$, either $\mathcal{S}_{D,G} = \emptyset$ or*

$$\max_{\mathcal{A} \in \mathcal{S}_{D,G}} \nu_k(V \setminus G) < \frac{1}{k}|V| \cdot \tilde{\phi}(k-1)$$

Proof. In any decision sequence D, the total vote weight of G in period k is $|G| \cdot \tilde{\phi}(k-1) \geq \frac{1}{k}|V| \cdot \tilde{\phi}(k-1)$, since each $v \in G$ has suffered $k-1$ consecutive losses. Thus, if the inequality holds, $\nu(G) > \nu(V \setminus G)$ and some candidate approved by a voter in G is guaranteed to win in period k. If it does not hold, then G can fail to be represented in period k (and therefore all periods $1, ..., k$) if all non-group voters $V \setminus G$ approve some other candidate in period k. □

Thus, given k-decision sequence D and group G, a logical approach for the adversary is to find an adversarial strategy $\mathcal{A} \in \mathcal{S}_{D,G}$ that maximizes $\nu_k(V \setminus G)$, the total vote weight of non-group voters in period k. This approach entails the following maximization

$$\max_{\mathcal{A} \in \mathcal{S}_{D,G}} \nu_k(V \setminus G) = \sum_{v \in V \setminus G} \alpha_k(v) \text{ subject to}$$
$$\frac{1}{i}|V| \cdot \tilde{\phi}(i-1) \leq \sum_{v \in (V \setminus G) \cap W_i} \alpha_i(v)$$

which amounts to choosing the number of non-group voters in each period that approve the winner among each equivalence class of vote weights. This problem reduces from an integer linear program whose behavior may change depending on the total number of voters $|V|$, and is NP-hard. Fortunately, we can make evaluating Lemma 5.1 easier by recognizing that integer-valued adversarial strategies exist in the space of real-valued adversarial strategies, and completing $\mathcal{S}_{D,G}$ to allow approval sets to be assigned to arbitrary "slices" of voters. A finite voter set restricts the actions of the adversary because it can only select integer quantities of voters to have particular preference profiles. If we can upper bound $\nu_k(V \setminus G)$ when the adversary can subdivide voters arbitrarily in a continuous setting, then this bound holds when the adversary must subdivide voters into integer quantities.

The "continuous electorate" requires some additional notation. For simplicity, we normalize the electorate to the interval $V = [0, k]$, and discuss vote "density" α_i in a way that is analogous to vote weight. For slice $V' \subset V$, we denote $\mu(V')$ the Lebesgue measure of V', which is invariant between periods; $\mu(V')/k$ corresponds to the slice's proportion of the total electorate. We distinguish between "voters" $\mu(V')$ and "votes" $\nu_i(V') = \int_{V',\mu} \alpha_i$, the weighted votes contributed by such voters in period i; for instance, if slice V' has uniform density α_i, then $\nu_i(V') = \mu(V') \cdot \alpha_i$.

Definition 5.2 (Ordering of strategies)**.** For a given D, G, adversarial strategy $\mathcal{A}$ *dominates* $\mathcal{A}'$ if applying $\mathcal{A}$ weakly increases $\nu_k(V \setminus G)$ compared to applying $\mathcal{A}'$. The *maximal strategy* $\mathcal{A} \in \mathcal{S}_{D,G}$ dominates all $\mathcal{A}' \in \mathcal{S}_{D,G}$.

In each period i, to deny G representation, the adversary must select enough voters to form the W_i that approves some alternative to G's supported candidate that wins the election. It is clear that since voters are never rewarded for approving additional winners, a maximal strategy entails selecting no more than exactly enough voters to beat G's candidate.

Definition 5.3. Assume applying strategy $\mathcal{A}$ produces winner sequence $w_{1:k-1}$. $\mathcal{A}$ is *Pareto efficient* in period i if (1) for all $v \in V \setminus G$, either $A_i(v) = \{w_i\}$ or $A_i(v) = \emptyset$; and (2) $\nu_i(W_i) = \nu_i(G)$.

Lemma 5.2. *If Φ is an ordinal-preserving rule*[6] *and $\mathcal{A}$ is an adversarial strategy, for any i, there exists a strategy $\mathcal{A}'$ such that $\mathcal{A}'$ dominates $\mathcal{A}$; $\mathcal{A} = \mathcal{A}'$ in periods $1, ..., i-1$; and $\mathcal{A}$ is Pareto efficient in period i. For any D, G, there exists a maximal strategy that is Pareto efficient in all periods.*

However, the optimal allocation of non-group voters to W_i depends on the particular rule – specifically, it depends on the direction of the following inequality.

Inequality 5.1. *Given $\tilde{\phi}$, define $\psi(0) = 0$ and $\psi(x) = \tilde{\phi}(x-1)$ for $x \in \mathbb{N}$. For all $k \in \mathbb{N}$, $m < n \in \mathbb{N}$:*

$$\frac{\psi(k+m) - \psi(k)}{\psi(m)} \ (\leq, \geq, =) \ \frac{\psi(k+n) - \psi(k)}{\psi(n)}$$

5.2 Discount rules

We briefly return to another consequence of Corollary 4.1:

Proposition 5.1. *Let $\tilde{\phi}$ define a consecutive-loss rule. If there exists $K, \varepsilon > 0$ such that for $k > K$, $\tilde{\phi}(k) \leq 2 \cdot \tilde{\phi}(k-1) - \varepsilon$, the rule fails JR.*

This limiting condition on consecutive-loss rules that satisfy JR motivates us to study the following class of rules:

Definition 5.4 (Discount rules)**.** For parameters δ, b, define the *discount* rule $\tilde{\phi}(k) = \delta \cdot \tilde{\phi}(k-1) + b$.

Theorem 5.1. *Discount rules with $\delta > 2.619$ and $b = 0$ satisfy JR for every way of breaking ties.*

The proof applies the following lemmas:

Lemma 5.3. *If $\tilde{\phi}$ is a discount rule with $b = 0$, then $\tilde{\phi}$ satisfies Inequality 5.1 in the $\leq$ direction.*

Lemma 5.4. *If $\tilde{\phi}$ satisfies Inequality 5.1 in the $\leq$ direction, then for all D, G over k periods, there exists a Pareto efficient maximal strategy under which for all $i \in [k]$, $W_i \subset W_{i+1}$.*

In other words, the maximal strategy is to preferentially select the non-group voters with the least density to approve the winner in every period.

Proof of Theorem 5.1. WLOG let $G = [0, 1]$. Consider the maximal strategy provided by Lemma 5.4. Since $W_{i-1} \subset W_i$ and $\nu_i(W_i) = \tilde{\phi}(i-1) = \delta^{i-1}$ for each i,

$$\mu(W_i) = 1 + \left(1 - \frac{1}{\delta^{i-1}}\right)\mu(W_{i-1}) = i - \sum_{j=1}^{i-1} \frac{\mu(W_j)}{\delta^j}$$

Now we try to bound $\nu_k(V \setminus G)$. We can loosely upper bound $\mu(W_i)$ by i:

$$\mu(W_{k-1}) \geq k - 1 - \sum_{i=1}^{m-2} \frac{i}{\delta^i} > m - 1 - \frac{\delta}{(1-\delta)^2}$$

$$\nu_m(V \setminus G) = \mu(W_{m-1}) + \delta^{m-1}(m - 1 - \mu(W_{m-1}))$$

Applying Lemma 5.1, a discount rule satisfies JR if we can show that under the maximal strategy, $\nu_k(V \setminus G) < \delta^{k-1} \leq \nu_k(G)$, which is the condition that

$$h_\delta(k) = \frac{\delta}{(1-\delta)^2} + \frac{k-1}{\delta^{k-1}} < 1$$

[6]All consecutive-loss rules are ordinal-preserving by definition.

For $\delta > 2.95$, the above holds for all $k \geq 3$. A technical note in Appendix B reduces the bound to $\delta > 2.619$. $\square$

Since higher δ causes more weight distortion between voters with the highest and lowest weights, we may wish to find the smallest δ for which a discount rule exists that satisfies JR. By setting $b = 1$, we can exactly achieve the theoretical lower bound given by Proposition 5.1.

Theorem 5.2. *Discount rules with $\delta \geq 2$ and $b = 1$ satisfy JR for every way of breaking ties.*

Again, we require the following lemmas:

Lemma 5.5. *If $\tilde{\phi}$ is a discount rule with $b = 1$, then $\tilde{\phi}$ satisfies Inequality 5.1 with exact equality (=).*

Lemma 5.6. *If $\tilde{\phi}$ satisfies Inequality 5.1 with exact equality, then for all D, G over k periods, all Pareto efficient strategies are maximal.*

Proof of Theorem 5.2. Let $G = [0, 1]$. Consider the following strategy: $W_i = [i, i+1]$ for $i \in [k-1]$. Since this strategy is Pareto efficient, by Lemma 5.6, it maximizes $\nu_k(V \setminus G)$. In period k, for each $i \in [k-1]$, $\alpha_i([i, i+1]) = \tilde{\phi}(k-i-1) = \sum_{j=0}^{k-i-1} \delta^j = \frac{\delta^{k-i}-1}{\delta-1}$. Thus, the JR condition is

$$\nu_k(V \setminus G) = \frac{\delta^k - kx + k - 1}{(\delta-1)^2} < \frac{\delta^k - 1}{\delta - 1} = \nu_k(G)$$

which is satisfied for all k iff $\delta \geq 2$. $\square$

5.3 Polynomial rules

While discount rules can satisfy JR, they do not satisfy EJR. For rules that do not provide an EJR guarantee, we can ask if rules satisfy an approximation to it:

Definition 5.5 (Approximations)**.** For a given k-decision sequence D, a winner sequence $\overline{w} = w_{1:k}$ α-approximates JR if for every cohesive $V' \subset V$ such that $|V'| > \alpha \cdot |V|/k$, there exists some period j in which $w_j \in \bigcup_{v \in V'} A_j(v)$. A winner sequence $\overline{w} = w_{1:k}$ α-approximates EJR if for every cohesive $V' \subset V$ such that $|V'| > \alpha\ell \cdot |V|/k$, it holds that for some voter $v \in V'$, $|\{i \in [k] \mid w_i \in A_i(v)\}| \geq \ell$. A voting rule α-approximates (E)JR if for every $k \in \mathbb{N}$, applying the rule to any k-decision sequence produces a winner sequence that α-approximates (E)JR.

Unfortunately, discount rules perform poorly even on this weaker metric.

Theorem 5.3. *If there exists $K, \delta > 1$ such that for $k > K$, $\tilde{\phi}(k) \geq \delta \cdot \tilde{\phi}(k-1)$, the rule fails any α-approximation of EJR for any way of breaking ties.*

Discount rules increase the weights of unsatisfied voters very rapidly, which enables them to satisfy JR because unsatisfied groups that have waited enough periods are guaranteed high enough weight to decide the election, but this weight distortion does not favor the rules' proportionality guarantees, since groups that have recently been represented are very heavily penalized. For rules that approximate both JR and EJR, we look to consecutive-loss rules with slower rates of increase.

Definition 5.6 (Polynomial)**.** For parameter p, define the *polynomial* rule $\tilde{\phi}(k) = (k+1)^p$.

Theorem 5.4. *The polynomial (linear) rule for $p = 1$ satisfies a 2-approximation of JR.*

The α-approximation proof for JR is the same as that for exact satisfaction: we assume $|G| \geq \alpha \cdot |V|/k$, and maximize $\nu_k(V \setminus G)$ over the space of adversarial strategies. The linear rule satisfies Inequality 5.1 with equality, so we apply Lemma 5.6 and the proof follows similarly to that of Theorem 5.2.

Theorem 5.5. *Polynomial rules for $1.01 < p < 10$ satisfy a $(1 + 1/p)$-approximation of JR.*

The proof uses the following lemmas, which state that the maximal strategy is to preferentially select the non-group voters with the most density to approve the winner in every period (unlike in Theorem 5.1).

Lemma 5.7. *If $\tilde{\phi}$ is a polynomial rule $p > 1$, then $\tilde{\phi}$ satisfies Inequality 5.1 in the $\geq$ direction.*

Lemma 5.8. *If $\tilde{\phi}$ satisfies Inequality 5.1 in the $\geq$ direction, then for all D, G over k periods, there exists a Pareto efficient maximal strategy under which for all $i \in [k]$, for any $A \subset W_i$ and $B \subset (V \setminus G) \setminus W_i$, $\frac{\nu_i(A)}{\mu(A)} \geq \frac{\nu_i(B)}{\mu(B)}$.*

The full proof of Theorem 5.5 is provided in Appendix B. Since polynomial rules increase voter weights slower than discount rules, they allow for approximation of EJR:

Theorem 5.6. *The polynomial (linear) rule for $p = 1$ satisfies a 2-approximation of EJR.*

Proof sketch. To prove 2-approximation of EJR, we need to extend our conception of adversarial strategies. Consider a k-decision instance D, and a cohesive group G, $|G| \geq 2\ell \cdot |V|/k$. Since any voter in G may be allowed to be satisfied up to $\ell - 1$ times without the winner sequence satisfying EJR, we allow the adversary to

- Partition G into finitely many subsets G_q;
- On each G_q, select the periods $R_q \subset [k-1]$ in which G_q is satisfied, such that $|R_q| \leq \ell - 1$;
- Choose the preferences of non-group voters $V \setminus G$ such that each G_q is satisfied in exactly periods R_q;

performing this process in a way that maximizes $\nu_k(V \setminus G) - \nu_k(G)$. The third step of this optimization follows the same maximization procedure as the JR adversary; since $\nu_k(G)$ depends only on the subsets G_q and the periods in which they are satisfied (R_q), the adversary must only choose how to allocate voters $V \setminus G$ to the W_i such that $\nu_k(V \setminus G)$ is maximized. Again, as the linear rule satisfies Inequality 5.1 with equality, a version of Lemma 5.6 holds: considering the first two steps of this optimization fixed, every Pareto optimal strategy on the third step is maximal.

For each i, define $H_i = G \setminus W_i$. Now for every period i, under any Pareto optimal strategy

$$\nu_{i+1}(V \setminus G) = \nu_i(V \setminus G) + \mu(V \setminus G) - \nu_i(H_i)$$

$$\nu_k(V \setminus G) = k \cdot \mu(V \setminus G) - \sum_{i=1}^{k-1} \nu_i(H_i)$$

Also, observe that

$$\nu_k(G) + \sum_{i=1}^{k-1} \nu_i(G \setminus H_i) = k \cdot \mu(G)$$

Since every voter in G is represented at most $\ell - 1$ times

$$\sum_{i=1}^{k-1} \nu_i(G) \geq \ell \cdot \frac{1}{2} \cdot \frac{k}{\ell} \cdot (\frac{k}{\ell} + 1)\mu(G) \geq \frac{k^2}{2\ell}\mu(G)$$

$$\begin{aligned}
&\nu_k(V \setminus G) - \nu_k(G) \\
&= k \cdot \mu(V \setminus G) - k \cdot \mu(G) - \sum_{i=1}^{k-1} \nu_i(G) \\
&\leq k(\frac{k}{2\ell})\mu(G) - k \cdot \mu(G) - \frac{k^2}{2\ell}\mu(G) < 0
\end{aligned}$$

so G must be represented in period k.

Remark 5.1. *Polynomial rules for $p > 1$ satisfy bounded approximations of EJR for $k \leq 100$ periods.*

We search among adversarial strategies to derive EJR guarantees for polynomial rules in settings with few periods. The results are described in further detail in Appendix C.

6 Simulations

While we have analyzed the theoretical worst-case (E)JR guarantees of perpetual voting rules, these rules may perform differently in practice if adversarial cases are sparse. Indeed, Bredereck et al [2019] found that under a range of randomized voting simulations analyzing single-period committee selection that a majority of randomly-selected committees satisfied (E)JR. We thus perform simulations to better understand the empirical performance of the rules discussed.

We compare several voting rules. As a baseline, we consider plain approval voting (AV), which does not take into account the perpetual setting. We consider the discount rules $\text{Discount}(\delta, b)$ for $(\delta, b) \in \{(2.5, 0), (1.5, 1), (2, 1)\}$, and the polynomial rules $\text{Poly}(n)$ for $n \in \{1, 1.1, 1.5, 2\}$. We add two win-based rules: GreedyCC from Bulteau et al [2021], and Perpetual PAV from Lackner and Maly [2023]. Finally, we add Perpetual Phragmen, Lackner and Maly [2023]'s preferred voting rule, and Perpetual Equal Shares. More information about these rules can be found in Appendix D.

For all simulations, we consider perpetual voting instances with N voters, M candidates, and k periods. We generate k-decision sequences as follows. We create voter preferences for the first election by, for each voter, randomly selecting approved candidates, with $M/4$ approvals on average. We generate subsequent preferences iteratively from previous election preferences: Each voter changes their votes for $\text{Geom}(0.5)$ arbitrarily-selected candidates, where Geom denotes the geometric distribution distributed on $\mathbb{Z}^{\geq 0}$. We discuss these design decisions further in Appendix D.

Since identifying EJR groups is NP-hard and thus computationally intractable for large simulations, we evaluate rules' satisfaction of EJR by explicitly constructing a cohesive group with distinct voting preferences, similar to a minority bloc, and testing for its representation. We consider $\ell \in \{1, 2, 3, 4\}$. For each value of ℓ, we consider instances with $M \in \{3, 4, 5\}$, $N = 1000$, and $k \in \{10, 20\}$. (Larger values of M, k, and ℓ were also tested, with similar results to those presented.) In each instance, after generating a k-decision sequence, we construct the group by randomly selecting a proportion of ℓ/k of the voters, and having them approve a randomly-selected candidate in each period, which other voters do not approve. We then evaluate if the winners produced by each voting rule fulfill the group's EJR guarantee. We run 100 instances for each setting of the parameters (totally 600 per value of ℓ) and calculate the percentage of all instances in which each rule produced a winner sequence that satisfies EJR. Results are summarized in Table 1.

ℓ	EJR Results for Rules
1	**AV** satisfied EJR in 0.2% of instances **Perpetual PAV** satisfied EJR in 90.0% of instances $\text{Poly}(1)$ satisfied EJR in 97.8% of instances All other rules satisfied EJR in 100% of instances
2	**AV** satisfied EJR in 0.0% of instances **Perpetual PAV** satisfied EJR in 92.7% of instances **GreedyCC** satisfied EJR in 46.2% of instances $\text{Discount}(2.5, 0)$ satisfied EJR in 99.7% of instances All other rules satisfied EJR in 100% of instances
3	**AV** satisfied EJR in 0.2% of instances **Perpetual PAV** satisfied EJR in 96.3% of instances **GreedyCC** satisfied EJR in 31.3% of instances $\text{Discount}(2.5, 0)$ satisfied EJR in 50.5% of instances $\text{Discount}(2, 1)$ satisfied EJR in 75.0% of instances All other rules satisfied EJR in 100% of instances
4	**AV** satisfied EJR in 50.0% of instances **Perpetual PAV** satisfied EJR in 97.2% of instances **GreedyCC** satisfied EJR in 19.7% of instances $\text{Discount}(2.5, 0)$ satisfied EJR in 2.0% of instances $\text{Discount}(2, 1)$ satisfied EJR in 18.7% of instances $\text{Discount}(1.5, 1)$ satisfied EJR in 94.2% of instances $\text{Poly}(2)$ satisfied EJR in 94.8% of instances $\text{Poly}(1.5)$ satisfied EJR in 99.8% of instances All other rules satisfied EJR in 100% of instances

Table 1: **Testing EJR on perpetual voting rules.** For each value of ℓ, we consider $N = 1000$ voters and vary M, k, resulting in 600 instances. Constructing a group of cohesive voters, we report the total percentage of instances where the winner sequence satisfies the group's EJR guarantee, i.e. where the group was successfully represented via EJR in the winner sequence produced by each rule.

An immediate observation is the necessity of having rules equipped for perpetual voting: AV fails prominently, satisfying EJR in virtually no instances, except for the case $\ell = 4$ (where we surmise that groups become large enough to possibly produce the majority vote).

Similarly, these results illustrate the advantage that worst-case guarantees provide. While GreedyCC performs well for $\ell = 1$ (where JR and EJR are equivalent), its performance degrades when considering larger groups. Perpetual Equal Shares satisfies EJR in all instances as expected. Methods which approximate JR or EJR also do well, which suggests that theoretical guarantees for approximations or weaker notions of representation can carry over to strong empirical performance. Discount rules performed well for $\ell = 1, 2$, with $\text{Discount}(1.5, 1)$ satisfying the majority of instances across all ℓ, while polynomial rules did even better. We also note that smaller values of δ in discount rules and smaller values of n in polynomial rules improved performance: the $\text{Poly}(1.1)$ rule satisfied EJR for all instances. Overall, basic rules can do well in satisfying EJR, as illustrated by the discount, polynomial, and Perpetual PAV rules. Here, their performance effectively matched that of Perpetual Phragmen and Equal Shares.

Additionally, rules satisfying EJR should intuitively demonstrate notions of proportionality: the larger a cohesive group, the more satisfied its voters should be (specifically, the maximum satisfaction of the group should increase). To explore this idea, we again consider the methodology for explicit group construction as mentioned above, but vary the size of the constructed group. Setting $M = 10$, $N = 100$, and $k = 50$, we consider cohesive group sizes within $[2, 40]$. For each size, we generate 100 instances and determine the maximum voter satisfaction among the group given the winners from the voting rule. Plotting the satisfaction against the group size (as a proportion of N) shows that rules which carry theoretical EJR guarantees or performed well above exhibit a stronger linear relationship between group size and maximum satisfaction, validating our intuition.

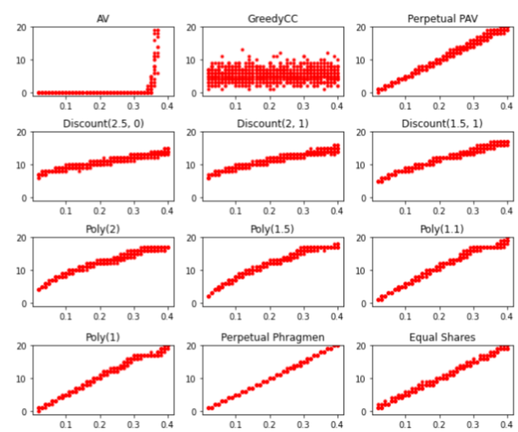


Figure 1: **Maximum Group Voter Satisfaction (vertical axis) versus Group Size (horizontal axis).** Rules which satisfied EJR poorly in Table 1 show little to no linearity between maximum satisfaction and size; better-performing methods exhibit stronger linearity.

7 Further research

This paper showed that interpretable weighted voting rules can satisfy (E)JR in perpetual voting. A few natural directions for further research arise.

First, since we treated the least complex basic perpetual rules as a starting point, a clear extension is to study rules that are one level up in complexity. For instance, rules that consider *both* the total number of wins and consecutive losses create $O(k^2)$ equivalence classes of voters after k periods. Further work may also consider other polynomial-complexity functions from $\mathcal{P}([k])$. One open question is whether more complex basic rules may approximate JR/EJR better than linearly bounded rules.

Furthermore, it may be interesting to consider versions of JR/EJR that consider cohesion in a subset of periods. Bulteau et al [2021] suggests "some-periods" versions of JR/EJR, but it is unclear whether any perpetual rule can be constructed to satisfy this criteria with an unknown time horizon. A perhaps more tractable choice is to fix a window length k, and evaluate whether a mechanism can be constructed that guarantees "all-periods" JR/EJR at any point in time considering only the most recent k elections.

References

[Aziz *et al.*, 2015] Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. Justified representation in approval-based committee voting. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence*, AAAI'15, page 784–790. AAAI Press, 2015.

[Aziz *et al.*, 2017] Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. Justified representation in approval-based committee voting. *Social Choice and Welfare*, 48(2):461–485, Feb 2017.

[Aziz *et al.*, 2018] Haris Aziz, Edith Elkind, Shenwei Huang, Martin Lackner, Luis Sanchez-Fernandez, and Piotr Skowron. On the complexity of extended and proportional justified representation. *Proceedings of the AAAI Conference on Artificial Intelligence*, 32(1), Apr. 2018.

[Bredereck *et al.*, 2019] Robert Bredereck, Piotr Faliszewski, Andrzej Kaczmarczyk, and Rolf Niedermeier. An experimental view on committees providing justified representation. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*, pages 109–115. International Joint Conferences on Artificial Intelligence Organization, 7 2019.

[Brill *et al.*, 2017] Markus Brill, Rupert Freeman, Svante Janson, and Martin Lackner. Phragmén's voting methods and justified representation. *Proceedings of the AAAI Conference on Artificial Intelligence*, 31(1), Feb. 2017.

[Brill *et al.*, 2020] Markus Brill, Paul Gölz, Dominik Peters, Ulrike Schmidt-Kraepelin, and Kai Wilker. Approval-based apportionment. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(02):1854–1861, Apr. 2020.

[Brill *et al.*, 2022a] Markus Brill, Paul Gölz, Dominik Peters, Ulrike Schmidt-Kraepelin, and Kai Wilker. Approval-based apportionment. *Mathematical Programming*, Jul 2022.

[Brill *et al.*, 2022b] Markus Brill, Jonas Israel, Evi Micha, and Jannik Peters. Individual representation in approval-based committee voting. In *Thirty-Sixth AAAI Conference on Artificial Intelligence, AAAI 2022, Thirty-Fourth Conference on Innovative Applications of Artificial Intelligence, IAAI 2022, The Twelveth Symposium on Educational Advances in Artificial Intelligence, EAAI 2022 Virtual Event, February 22 - March 1, 2022*, pages 4892–4899. AAAI Press, 2022.

[Bulteau *et al.*, 2021] Laurent Bulteau, Noam Hazon, Rutvik Page, Ariel Rosenfeld, and Nimrod Talmon. Justified representation for perpetual voting. *IEEE Access*, 9:96598–96612, 2021.

[Casella, 2012] Alessandra Casella. *Storable Votes: Protecting the Minority Voice*. Number 9780195309089 in OUP Catalogue. Oxford University Press, November 2012.

[Do *et al.*, 2022] Virginie Do, Matthieu Hervouin, Jérôme Lang, and Piotr Skowron. Online approval committee elections. In Luc De Raedt, editor, *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022, Vienna, Austria, 23-29 July 2022*, pages 251–257. ijcai.org, 2022.

[Freeman and Conitzer, 2016] Rupert Freeman and Vincent Conitzer. Nash welfare and round robin share. *CoRR*, abs/1611.04034, 2016.

[Freeman *et al.*, 2017] Rupert Freeman, Seyed Majid Zahedi, and Vincent Conitzer. Fair and efficient social choice in dynamic settings. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*, pages 4580–4587, 2017.

[Harrenstein *et al.*, 2020] Paul Harrenstein, Marie-Louise Lackner, and Martin Lackner. A mathematical analysis of an election system proposed by gottlob frege. *Erkenntnis*, 87:1–36, 11 2020.

[Imber *et al.*, 2022] Aviram Imber, Jonas Israel, Markus Brill, and Benny Kimelfeld. Approval-based committee voting under incomplete information. In *Thirty-Sixth AAAI Conference on Artificial Intelligence, AAAI 2022, Thirty-Fourth Conference on Innovative Applications of Artificial Intelligence, IAAI 2022, The Twelveth Symposium on Educational Advances in Artificial Intelligence, EAAI 2022 Virtual Event, February 22 - March 1, 2022*, pages 5076–5083. AAAI Press, 2022.

[Israel and Brill, 2021] Jonas Israel and Markus Brill. Dynamic proportional rankings. In Zhi-Hua Zhou, editor, *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021*, pages 261–267. ijcai.org, 2021.

[Kilgour, 2010] M. Kilgour. *Approval Balloting for Multi-Winner Elections*. Springer Berlin Heidelberg, 2010.

[Lackner and Maly, 2021] Martin Lackner and Jan Maly. Perpetual voting: The axiomatic lens. *CoRR*, abs/2104.15058, 2021.

[Lackner and Maly, 2023] Martin Lackner and Jan Maly. Proportional decisions in perpetual voting. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI 2023)*. AAAI Press, 2023.

[Lackner, 2020] Martin Lackner. Perpetual voting: Fairness in long-term decision making. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(02):2103–2110, Apr. 2020.

[Parkes and Procaccia, 2013] David Parkes and Ariel Procaccia. Dynamic social choice with evolving preferences. *Proceedings of the AAAI Conference on Artificial Intelligence*, 27(1):767–773, Jun. 2013.

[Peters and Skowron, 2020] Dominik Peters and Piotr Skowron. Proportionality and the limits of welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation*, EC '20, page 793–794, New York, NY, USA, 2020. Association for Computing Machinery.

[Sánchez-Fernández *et al.*, 2017] Luis Sánchez-Fernández, Edith Elkind, Martin Lackner, Norberto Fernández, Jesús

Fisteus, Pablo Basanta Val, and Piotr Skowron. Proportional justified representation. *Proceedings of the AAAI Conference on Artificial Intelligence*, 31(1), Feb. 2017.

[Thiele, 1895] T. N. Thiele. *Om flerfoldsvalg: Meddelt I Mødet Den 29. November 1895*. 1895.

A Additional notes

A.1 Footnote 1: Note on empty approval sets

Lackner [2020] does not allow voters to express the empty approval set. Indeed, one desideratum is that perpetual voting rules should elicit voters' true preferences, and one way to entice voters into expressing their true views in every period is to eliminate the option of overtly strategizing by approving nothing, which guarantees that they will be left unsatisfied and therefore gain influence in later elections. However, employing this restriction merely obfuscates the strategic value of purposefully attempting to approve no winning candidates to increase one's vote weight in the future, and in the general space of decision sequences when the number of candidates may exceed the number of voters, whether or not empty approvals are allowed makes no difference for JR and EJR properties. It is easy to observe this fact in the continuous-electorate settings we discuss in the later sections of the paper: if an adversary wishes to select a strategy such that voters $V' \subset V \setminus G$ are guaranteed to not have approved the winner, but still approve something in the electorate, we can simply subdivide V' into many infinitesimal slices such that the total vote weight of each slice is nearly zero, then declare all the slices to have approved disjoint sets of candidates. In the discrete setting, we can achieve a similar setup by having enough voters to mimic this infinitesimal subdivision (so that the vote weight of each individual slice in period i never exceeds $\nu_i(W_i)$).

We can see that this strategy does not work in more restricted settings when the number of candidates is bounded in each period, and small relative to the vote weights of individual voters. For instance, if the number of candidates must be $\leq M$ in every period, and voters are not allowed to approve the empty set, then the winning candidate must receive $\nu_i(W_i) \geq \frac{1}{M}\nu_i(V)$ weighted votes, which is a meaningful restriction on adversarial strategies. This setup may be an additional interesting direction for further research.

A.2 General JR test for basic rules

To show discount rules satisfy JR in Section 5, we proved that some optimal adversarial strategy maximizes $\mu(V \setminus G)$. However, for more complex rules, the optimal adversarial strategy may be hard to articulate. If the goal is to prove JR only up to some fixed number of periods k, it is possible to utilize a linear program to find the optimal value of $\nu_k(V \setminus G)$. For every $i \in [k-1]$, for every $R \subset [i]$, define variable $x_{i,R}$. Assuming a continuous electorate $[0, k]$, $x_{i,R} := \mu(\{v \in V \setminus G \mid R_i(v) = R\})$, where $R_i(v)$ is the win-loss record of voter v up to and including period i. We have the following constraints:

$$\max \sum_{R \subset [k-1]} \phi_k(R) x_{k-1,R} \text{ subject to}$$
$$x_{1,\{1\}} + x_{1,\emptyset} \leq k-1$$
$$\forall i > 1, \forall R \subset [i-1],\ x_{i,R\cup\{i\}} + x_{i,R} = x_{i-1,R}$$
$$\forall i,\ \sum_{R \subset [i-1]} \phi_i(R) \cdot x_{i,R\cup\{i\}} \geq \phi_i(\emptyset)$$

Since the number of constraints is exponential in the number of periods, using this linear program to test JR for large k is computationally infeasible. However, for rules with polynomial complexity functions, since the number of equivalence classes of voters grows polynomially with the number of periods, so does the number of constraints in this linear program, as there is one constraint for each equivalence class.

B Omitted proofs

Proofs appear in the order that statements are presented in the paper.

B.1 Section 2

Proposition 2.1

Assume k periods. Let $G \subset V$ be a cohesive group such that $|G| \geq \ell \cdot |V|/k$, and assume that every $v \in G$ is satisfied in at most $\ell - 1$ periods. Define $p := |V|/k$.

Define $B_{k+1}(v)$ to be a voter's budget *after* period k.

Lemma. After period k, some voter $v \in G$ must have budget $B_{k+1}(v) < p/|G|$.

Suppose the candidate is arbitrarily selected in period k. Then in some period $i \leq k$, there must exist no affordable candidates. If all voters $v \in G$ have a budget of at least $p/|G|$ in period k, they must have a budget of at least $p/|G|$ in period i, and since G is cohesive, they must agree on some candidate c in period i. Thus, c is approved by voters with combined budget at least $p/|G| \cdot |G| = p$, so c is therefore affordable in period i, which is a contradiction.

Suppose the candidate is not arbitrarily selected in period k. All voters start with a combined budget of $|V|$ and p is subtracted from the combined budget in every period, so after k periods, the combined budget of all voters is $|V| - k \cdot p = 0$, and therefore all voters $v \in G$ have budget 0. □

Consider such voter $v \in G$ whose budget $B_{k+1}(v) < p/|G|$. Since v approves at most $\ell - 1$ candidates in periods $i \in [k]$, in some period i, they must have paid strictly more than $\frac{1-p/|G|}{\ell-1} \geq \frac{1-1/\ell}{\ell-1} = \frac{1}{\ell}$.

Consider the first period i in which some voter $v \in G$ pays strictly more than $\frac{1}{\ell}$. Since voter v paid strictly more than $\frac{1}{\ell}$ for the winning candidate w_i, the q-value for w_i must be strictly greater than $\frac{1}{\ell}$. On the other hand, each voter $v' \in G$ approved $\leq \ell - 1$ candidates in periods $i' < i$ and paid no more than $\frac{1}{\ell}$ for them, so the budget of v' in period i is at least $\frac{1}{\ell}$. Furthermore, since G is cohesive, all $v' \in G$ agree on some candidate c in period i. Thus, in period i, the q-value for c is at most $\frac{1}{\ell}$ (since each $v' \in G$ paying $\frac{1}{\ell}$ allows paying candidate c's total cost of $|V|/k = \frac{1}{\ell} \cdot \ell \cdot |V|/k \leq \frac{1}{\ell} \cdot |G|$), which is a contradiction since the mechanism chooses the candidate with the lowest q-value. □

B.2 Section 3

Equivalence of Definitions 3.1 and 3.2

Suppose $B = (f_i, g_i)_{i\in\mathbb{N}}$ represents a basic rule under Definition 3.1.

Construct a basic rule Φ under definition 3.2 as follows. Define $\phi_1(\emptyset) = 1$. Assume ϕ_i has been defined; now we define ϕ_{i+1}. Consider some $R \subset [i]$. If $i \in R$, define $\phi_{i+1}(R) = g_i(\phi_i(R \setminus \{i\}))$. If $i \notin R$, define $\phi_{i+1}(R) = f_i(\phi_i(R))$.

Claim any voter v with some record R in period k has the same weight under rules B and Φ. In period 1, $\alpha_1(v) = \phi_1(\emptyset) = 1$. Assume that in period i, v has the same weight $\alpha_i(v) = \phi_i(R \cap [i-1])$ (since the record of v in period i is $R \cap [i-1]$). Then if $w_i \in A_i(v) \Leftrightarrow i \in R$, its weight in period $i+1$ is $g_i(\alpha_i(v)) = \phi_{i+1}(R \cap [i])$ by definition. If $w_i \notin A_i(v) \Leftrightarrow i \notin R$, its weight in period $i+1$ is $f_i(\alpha_i(v)) = \phi_{i+1}(R \cap [i])$ by definition.

To show the bulleted axioms:

- For any k and $R \subset [k-1]$, $\phi_{k+1}(R) = f_i(\phi_k(R)) \geq \phi_k(R)$. Similarly, $\phi_{k+1}(R \cup \{i\}) = g_i(\phi_k(R)) \leq \phi_k(R)$.
- If $\phi_k(R) = \phi_k(R')$, then $\phi_{k+1}(R) = f_i(\phi_k(R)) = f_i(\phi_k(R')) = \phi_{k+1}(R')$. Also, $\phi_{k+1}(R \cup \{i\}) = g_i(\phi_k(R)) = g_i(\phi_k(R')) = \phi_{k+1}(R' \cup \{k\})$.

For the other direction, suppose Φ is a basic rule under Definition 3.2. Define B as follows. For all i, for every $R \subset [i-1]$, let $f_i(\phi_i(R)) = \phi_{i+1}(R)$, and let $g_i(\phi_i(R)) = \phi_{i+1}(R \cup \{i\})$. Let $f_i(x) = x$ and $g_i(x) = x$ otherwise (these values of f_i and g_i do not matter). Observe that f_i and g_i are well-defined due to the second bulleted axiom in Definition 3.2: if $\phi_i(R) = \phi_i(R')$, then $\phi_{i+1}(R) = \phi_{i+1}(R')$ and $\phi_{i+1}(R \cup \{i\}) = \phi_{i+1}(R' \cup \{i\})$.

Voters' weights are equivalent under the two rules by the same argument as the forward direction. Also, observe that the first bulleted axiom in Definition 3.2 implies $f_i(x) \geq x$ and $g_i(x) \leq x$. □

B.3 Section 4

Proposition 4.1

Proof by contrapositive. Assume the condition in Proposition 4.1 is not met for some $k \in \mathbb{N}$. Consider the following k-decision sequence:

$$V = v_{1:k},\ C = c_{1:k}$$
$$\forall j < k,\ \forall j,\ A_j(v_j) = \{c_j\}$$
$$A_k(v_k) = \{c_k\};\ \forall j \neq k,\ A_k(v_j) = \{c_1\}$$

We will show that for some way of breaking ties, the rule produces winner sequence $w = (c_1, c_2, ..., c_{k-1}, c_1)$.

First, we claim inductively that for some way of breaking ties, $w_j = c_j$ for periods $j \in [k-1]$. For $j = 1$, each candidate c_j receives approval from only v_j and all voters have weight $\phi_1(\emptyset) = 1$, so all candidates receive 1 weighted vote and we can choose to break ties in favor of c_1. For the inductive step, let $i < k$ and assume $w_j = c_j$ for $j \in [i-1]$. Again, each candidate c_j receives approval from only v_j. In period i, for $j < i$, v_j approved $c_j = w_j$ in period j, so they have weight $\phi_i(\{j\})$. For $j \geq i$, v_j did not approve a winner in prior periods, so they have weight $\phi_i(\emptyset)$. Candidates c_1 through c_{i-1} therefore receive $\phi_i(\{j\})$ weighted votes, and candidates c_i through c_k receive $\phi_i(\emptyset) \geq \phi_i(\{j\})$ weighted votes, so we can choose to break ties in favor of c_i. Thus $w_i = c_i$.

In period k, c_1 receives approval from $v_1, ..., v_{k-1}$. The win/loss record of each v_j is $\{j\}$, so c_1 receives $\sum_{j=1}^{k-1} \phi_k(\{j\})$ weighted votes, while c_k receives only the approval of v_k with $\phi_k(\emptyset)$ weighted votes. $c_2, ..., c_{k-1}$ receive 0 votes. Since the condition in the lemma is not met, c_1 receives equal or greater weighted votes than c_k, and we can break ties in favor of c_1.

The sequence $w = (c_1, c_2, ..., c_{k-1}, c_1)$ fails JR because while $|\{v_k\}| = 1 \geq k/k = |V|/k$, and $A_j(v_k) > 0$ for all j, v_k is not satisfied in any period. □

Corollary 4.1

Follow the same setup as the proof of Proposition 4.1 up to period $k-1$. By the pidgeonhole principle, there exists at least one candidate $c_{j^*} \in \{c_1, ..., c_k\}$ not elected in periods 1 to $k-1$. If there exists more than one such j^*, the winner sequence does not satisfy JR regardless of the outcome in election k, so we can safely assume j^* is unique. Let $A_k(v_i) = c_1$ for all $i \neq j^*$, and let $A_k(v_{j^*}) = c_2$. To satisfy JR, c_2 must receive at least equal votes as c_1 in period k. □

Proposition 4.2

By Corollary 4.1, for JR to be satisfied, for all $k \in \mathbb{N}$

$$1 = \tilde{\phi}(0) = \phi_k(\emptyset) \geq \sum_{i=1}^{k-1} \phi_k(\{i\}) = \sum_{i=1}^{k-1} \tilde{\phi}(1) = (k-1)\tilde{\phi}(1)$$

If $\tilde{\phi}(1) > 0$, then the condition is violated for $k = \left\lceil \frac{1}{\tilde{\phi}(1)} \right\rceil + 1$. It follows that $\tilde{\phi}(k) = 0$ for all $k \geq 1$.

The converse of Proposition 4.2 also holds: the win-based rule with $\tilde{\phi}(k) = 0$ for all $k \geq 1$ is the "Greedy Chamberain-Courant" rule (Greedy-CC), which satisfies JR as noted in Bulteau et al [2021]. □

Proposition 4.3

By Corollary 4.1, for JR to be satisfied, for all j

$$\tilde{\phi}(j) = \phi_{j+1}(\emptyset) \geq \sum_{i=1}^{j} \phi_{j+1}(\{i\}) = j \cdot \tilde{\phi}(j-1)$$

If $\tilde{\phi}(k) < k!$, then for some $j \leq k$, $\tilde{\phi}(j) < j \cdot \tilde{\phi}(j-1)$. □

Proposition 4.4

By Corollary 4.1, for JR to be satisfied, for all $k \in \mathbb{N}$

$$1 = \tilde{\phi}(0) = \phi_k(\emptyset) \geq \sum_{i=1}^{k-1} \phi_k(\{i\})$$
$$= \tilde{\phi}(1) + \sum_{i=1}^{k-2} \tilde{\phi}(0) = \tilde{\phi}(1) + k - 2$$

For $k \geq 3$, this is violated. □

Proposition 4.5

As JR is necessary for EJR, we need only show that the exceptions found in Propositions 4.2 and 4.3 do not satisfy EJR.

For Proposition 4.2: Greedy-CC does not satisfy EJR.

Proof by counterexample. Consider a 6-voter electorate $V = v_{1:6}$ over 4 periods. Assume candidates and preferences are static and v_1 always approves only c_1, v_2 always approves only c_2, v_3 always approves only c_3, and $\{v_4, v_5, v_6\}$ approves only c_4. $G = \{v_4, v_5, v_6\}$ form a cohesive group that deserves 2 periods of representation. However, in period 1, c_4 wins and the weights of v_4, v_5, v_6 are set to zero, so in periods 2,3,4 the other candidates win in some order, and no voter in G is satisfied more than once.

For Proposition 4.3: Loss-based rules with $\tilde{\phi}(k) \geq k!$ for all k do not satisfy EJR.

Proof by counterexample. Let r_m map to the total number of losses and $\tilde{\phi}(k) = k!$. Consider the following 6-decision sequence with 6 voters and 2 candidates in each period. The table shows the voters' weights in each period with the voter's number of losses in parentheses, where (*) denotes that the voter's approval set is $\{c_1\}$ and ($\sim$) denotes that the voter's approval set is $\{c_2\}$.

Period	v_1	v_2	v_3	v_4	v_5	v_6	Winner
1	1*	1*	1*	1	1	1	c_1
2	1*	1*	1*	1	1	1	c_1
3	1*	1*	1*	2$\sim$	2$\sim$	2	c_2
4	1*	1*	1*	2	2	6$\sim$	c_2
5	2*	2*	2*	6$\sim$	6$\sim$	6	c_2
6	6*	6*	6*	6	6	24$\sim$	c_2

The following table indicates the losses of each voter in the above example.

Period	v_1	v_2	v_3	v_4	v_5	v_6	Winner
1	0	0	0	0	0	0	c_1
2	0	0	0	1	1	1	c_1
3	0	0	0	2	2	2	c_2
4	1	1	1	2	2	3	c_2
5	2	2	2	3	3	3	c_2
6	3	3	3	3	3	4	c_2

$G = \{v_1, v_2, v_3\}$ is cohesive and deserves 3 periods of representation, but only gets 2. The method of constructing counterexamples for $\tilde{\phi}(k) > k!$ follows similarly: the weight of voters with a single fewer loss increases too fast to achieve proportionality. ☐

B.4 Section 5.1: Adversarial strategies

Lemma 5.2

Let $\mathcal{A}' = \mathcal{A}$ in periods $1, ..., i-1$. Start with $\mathcal{A}' = A$ in period i.

In period i, if for some $v \in V \setminus G$, $c \neq w_i$, and $c \in A_i(v)$, removing c from $A_i(v)$ does not change previous winners or the winner in period i, since c already receives fewer votes than w_i. Furthermore, since removing c does not change the win/loss record of any voter in period i, it does not affect vote weights or winners in any future period and therefore does not affect $\nu_k(V \setminus G)$. Update $\mathcal{A}'$ by removing c from $A_i(v)$. Iteratively updating $\mathcal{A}'$ over all $v \in V \setminus G$ gives us a strategy that satisfies the first axiom of Pareto efficiency.

Since G approves some candidate in period i and such candidate does not win in period i because $\mathcal{A}$ is an adversarial strategy, $\nu_i(W_i) \geq \nu_i(G)$. If $\nu_i(W_i) = \nu_i(G)$, we are done. Assume $\nu_i(W_i) > \nu_i(G)$. There exists $X \subset W_i$ such that $\nu_i(W_i \setminus X) = \nu_i(G)$.

Update $\mathcal{A}'$ by letting $A_i(X) = \emptyset$. Observe that winners are fixed, and for periods $i' < i$, no preferences change. In period i, w_i receives $\nu_i(G)$ votes; candidates approved by some voter in G receive at most $\nu_i(G)$ votes; other candidates receive no votes. Thus the winner w_i is preserved. For periods $i' > i$, for $x \in X$, the win/loss record R_x is only changed by removing period i, and by ordinal preservation, $\phi_{i'}(R_x \setminus \{i\}) \geq \phi_{i'}(R_x)$, so votes for $w_{i'}$ weakly increase. $\nu_k(X')$ also weakly increases, and since winners are fixed, $\nu_k(V \setminus (G \cup X'))$ is fixed, so $\nu_k(V \setminus G)$ weakly increases. Now, $\mathcal{A}'$ dominates $\mathcal{A}$, equals $\mathcal{A}$ in periods $1, ..., i-1$, and is Pareto efficient in period i.

For the claim about the maximal strategy: since we can write down the maximization of $\nu_k(V \setminus G)$ as a linear program, there exists a maximal strategy. To see that there exists a Pareto efficient maximal strategy, notice that given any adversarial strategy $\mathcal{A}$, we can iterate the above procedure over all periods, producing a strategy $\mathcal{A}'$ that dominates $\mathcal{A}$ and is Pareto efficient in all periods. As a result, some strategy that is Pareto efficient in all periods must be maximal. ☐

B.5 Section 5.2: Discount rules

Lemma 5.3

Since $\tilde{\phi}(k) = \delta \cdot \tilde{\phi}(k-1)$ and $\tilde{\phi}(0) = 1$, $\tilde{\phi}(k) = \delta^k$. Thus $\psi(k) = \delta^{k-1}$, and we have

$$\frac{\psi(k+m) - \psi(k)}{\psi(m)} = \frac{\delta^{k+m-1} - \delta^{k-1}}{\delta^{m-1}} = \delta^k - \delta^{k-m}$$

$$\frac{\psi(k+n) - \psi(k)}{\psi(n)} = \frac{\delta^{k+n-1} - \delta^{k-1}}{\delta^{n-1}} = \delta^k - \delta^{k-n}$$

For $m < n$,

$$\delta^k - \delta^{k-m} \leq \delta^k - \delta^{k-n}. \quad ☐$$

Lemma 5.4

Let $\mathcal{A}$ be some adversarial strategy. Using Lemma 5.2, we can find $\mathcal{A}_1$ that dominates $\mathcal{A}$ and is Pareto efficient in all periods.

We create a sequence of adversarial strategies $\mathcal{A}_1, ..., \mathcal{A}_k$ where the $\mathcal{A}_i$ are Pareto efficient in all periods, $\mathcal{A}_{i+1}$ dominates $\mathcal{A}_i$ for each i, and $\mathcal{A}_k$ is a Pareto efficient maximal strategy under which for all $i \in [k]$, $W_i \subset W_{i+1}$. (This property corresponds to nongroup voters with the least density approving the winner in every period.)

Consider strategy $\mathcal{A}_i$. Start with $\mathcal{A}_{i+1} = \mathcal{A}_i$. The general idea is that in the first period i in which the strategy is not allocatively efficient, we can identify some lower-density subset X that does not approve the winner and higher-density subset Y that does approve the winner and show that we can carefully swap the preferences of X and Y such that winners are preserved in every period and the vote weight $\nu_k(V \setminus G)$ weakly increases. We can continue iterating this procedure until $\mathcal{A}_{i+1}$ is allocatively efficient in period i.

Let i be the first period for which $\mathcal{A}_i$ violates the above property; in other words, the first period i in which $\mu(W_{i-1} \setminus W_i) > 0$. Then there must exist some subsets $X \subset W_i$, $Y \subset V \setminus W_i$ with uniform densities in period i such that X has a lower density than Y, and $\nu_i(X) = \nu_i(Y)$. WLOG we can assume X and Y each have uniform preferences for periods $i+1, ..., m-1$ (otherwise, since there are only a finite number of possible preferences, we can subdivide X, Y into subsets with uniform preferences, and consider these separately).

Suppose that in period i, X has density $\tilde{\phi}(m)$ and Y has density $\tilde{\phi}(n)$, with $m < n$.

Denote $R_X := \{i' > i \mid X \subset W_i\}$ and $R_Y := \{i' > i \mid Y \subset W_i\}$. Since $\mu(X) > \mu(Y)$, we can pick a subset $X' \subset X$ such that $\mu(Y) = \mu(X)$.

Update $\mathcal{A}_{i+1}$: let $A_i(X) = \{w_i\}$ and $A_i(Y) = \emptyset$, and for periods $i' > i$, let voters in X' and Y swap preferences. We want to show that winners are fixed.

- For periods $i' < i$, no preferences change.
- For period i, since $\nu_i(X) = \nu_i(Y)$, votes for w_i are fixed, and other candidates' votes are fixed.

Consider periods $i' > i$. Let $\nu'_{i'}$, $W'_{i'}$ denote the analogues of $\nu_{i'}$, $W_{i'}$ under the updated strategy. Consider each i' iteratively assuming previous winners are fixed. We will now show that the winner in period i' is fixed.

- If $i' \notin R_X \cup R_Y$, clearly $W_{i'} = W'_{i'}$ and $\nu'_{i'}(W'_{i'}) = \nu_{i'}(W_{i'})$, since neither X nor Y approve any candidates in either the old or updated strategies.
- If $i' = \min R_X$. Observe that
$$W'_{i'} = (W_{i'} \setminus X) \cup (Y \cup X \setminus X')$$
We want to show that $\nu'_{i'}(W'_{i'}) \geq \nu_{i'}(W_{i'})$. Since previous winners are fixed and no voters' preferences have changed other than those of X and Y,
$$\nu'_{i'}(W'_{i'}) = \nu_{i'}(W_{i'}) - \nu_{i'}(X) + \nu'_{i'}(Y \cup X \setminus X')$$
so it suffices to show $\nu_{i'}(X) \leq \nu'_{i'}(Y \cup X \setminus X')$. We derive this condition from Inequality 5.1.

  Let $k = i' - i$. In the old strategy, $\nu_{i'}(X) = \tilde{\phi}(m + k)\mu(X)$, and in the updated strategy, $\nu'_{i'}(Y) = \tilde{\phi}(n + k)\mu(Y)$ and $\nu'_{i'}(X \setminus X') = \tilde{\phi}(k-1)\mu(X \setminus X')$. Let ν denote $\nu_i(X) = \nu_i(Y)$, then $\mu(X) = \frac{\nu}{\tilde{\phi}(m)}$, $\mu(Y) = \frac{\nu}{\tilde{\phi}(n)}$, and $\mu(X \setminus X') = \frac{\nu}{\tilde{\phi}(m)} - \frac{\nu}{\tilde{\phi}(n)}$. Now the above condition is the following inequality:
$$\frac{\tilde{\phi}(m+k) - \tilde{\phi}(k-1)}{\tilde{\phi}(m)} \leq \frac{\tilde{\phi}(n+k) - \tilde{\phi}(k-1)}{\tilde{\phi}(n)}$$
which directly follows from Inequality 5.1.
- If $i' \in R_X \setminus \{\min R_X\}$, observe that since $\mu(X) = \mu(Y \cup X \setminus X')$ and consecutive losses prior to period i' for $Y \cup X \setminus X'$ in the updated strategy equal those for X in the old strategy, $\nu'_{i'}(Y \cup X \setminus X') = \nu_{i'}(X)$.
- If $i' \in R_Y$, the same statement holds for X' and Y. Thus all $w_{i'}$ receive weakly increased votes under the updated strategy.

The intuition is as follows: since $\mu(X') = \mu(Y)$, in any future periods in which Y previously approved the winner, the necessary vote weight can be taken care of by directing X' to approve. For future periods in which X previously approved the winner, it suffices to check that $Y \cup X \setminus X'$ (which has the same measure as X) can take care of the vote weight of X in the *first* period after i that X approves, which comes from Inequality 5.1; in later periods in which X previously approved, the vote weight of $Y \cup X \setminus X'$ was reset in the same period as X was previously, so the necessary vote weight can be taken care of by directing $Y \cup X \setminus X'$ to approve.

Note that after this update, $\nu_k(V \setminus G)$ weakly increases. Since winners are fixed, $\nu_k(V \setminus (G \cup X \cup Y))$ is fixed. In period k, consecutive losses for X' in the updated strategy equal those for Y in the old strategy, and $\mu(X') = \mu(Y)$, so $\nu'_k(X') = \nu_k(Y)$. If $R_X = \emptyset$, then again $\nu'_k(Y \cup X \setminus X') \geq \nu_k(X)$ by Inequality 5.1, and otherwise consecutive losses for $Y \cup X \setminus X'$ in the updated strategy equal those for X in the old strategy, so $\nu'_k(Y \cup X \setminus X') \geq \nu_k(X)$.

We can continue iterating these swaps until all the voters with the least density approve w_i in period i. Finally, we need to update strategy $\mathcal{A}_{i+1}$ so that it is Pareto efficient. We can iteratively apply Lemma 5.2 to obtain a strategy that equals the original $\mathcal{A}_{i+1}$ on all periods $1, ..., i$ and is Pareto efficient in the remaining periods. Now define $\mathcal{A}_{i+1}$ to be the next strategy in the sequence. Continuing the sequence until $\mathcal{A}_k$, we obtain a strategy $\mathcal{A}_k$ that dominates $\mathcal{A}$ and satisfies the condition in Lemma 5.4. Conclude that since any strategy $\mathcal{A}$ is dominated by a strategy that satisfies the condition in the lemma, there exists some maximal strategy satisfying the condition. □

Theorem 5.1 additional notes

Clarification: in the abridged version of the proof in the paper text, we made a typo substituting the wrong symbol m for the symbol k in two lines of this proof. We apologize for the error. For clarity, the entire proof is included here, including the corrected version of the part included in the paper text.

Proof of Theorem 5.1. WLOG let $G = [0, 1]$. Consider the maximal strategy provided by Lemma 5.4. Since $W_{i-1} \subset W_i$ and $\nu_i(W_i) = \tilde{\phi}(i-1) = \delta^{i-1}$ for each i,
$$\mu(W_i) = 1 + \Big(1 - \frac{1}{\delta^{i-1}}\Big)\mu(W_{i-1}) = i - \sum_{j=1}^{i-1} \frac{\mu(W_j)}{\delta^j}$$
Now we try to bound $\nu_k(V \setminus G)$. We can loosely upper bound $\mu(W_i)$ by i:
$$\mu(W_{k-1}) \geq k - 1 - \sum_{i=1}^{k-2} \frac{i}{\delta^i} > k - 1 - \frac{\delta}{(1-\delta)^2}$$
$$\nu_k(V \setminus G) = \mu(W_{k-1}) + \delta^{k-1}(k - 1 - \mu(W_{k-1}))$$
Applying Lemma 5.1, a discount rule satisfies JR if we can show that under the maximal strategy, $\nu_k(V \setminus G) < \delta^{k-1} \leq \nu_k(G)$, which is the condition that
$$h_\delta(k) = \frac{\delta}{(1-\delta)^2} + \frac{k-1}{\delta^{k-1}} < 1$$
For $\delta > 2.619$, there exists K for which $h_\delta(K) < 1$. Since $h'_\delta(k) < 0$ for $k > \sqrt{\delta} + 1$, $h_\delta(k) < 1$ for all $k \geq K$, which implies JR for $k \geq K$ periods.

Furthermore, we claim that JR for K periods implies JR for $k < K$ periods. Assume some discount rule with $b = 0$ fails JR for $k < K$ periods. Then, for $V = [0, K]$, $C = \{c, w_1, ..., w_K\}$. Assume the cohesive group $[0, 1]$ approves $\{c\}$ in every period. There exists some strategy for which in period k, $\nu_k([1, k]) \geq \tilde{\phi}(k-1) = \delta^{k-1}$ and $[0, 1] \cup [k, K]$ remains unrepresented. Then, for each $i \in \{k, ..., K-1\}$, let $W_i = [i-1, i]$ with $A_i(v) = \{w_i\}$ for $v \in W_i$ and $\emptyset$ for $v \in [1, K] \setminus W_K$. In period K, let $[1, K]$ approve $\{w_K\}$. $\nu_K([1, K]) > \nu_K([1, k]) = \delta^{K-k}\nu_k([1, k]) \geq \delta^{K-1} = \nu_K([0, 1])$, so w_K wins in period K and $[0, 1]$ is unrepresented in periods $i \in [K]$, failing JR for K periods. □

Since the proof of this theorem does not use a tight bound on $\mu(W_{k-1})$, calculating the number of non-group votes generated by the maximal adversarial strategy indicates a possibly lower optimal bound for δ.

Proposition. Discount rules with $\delta > 2.4622$ and $b = 0$ satisfy JR for $k \leq 100000$ periods. Consider the maximal adversarial strategy. For all i,

$$\mu(W_i) = 1 + (1 - \frac{1}{\delta^{i-1}})\mu(W_{i-1})$$

The votes of non-group voters in period k is

$$\nu_k(V \setminus G) = \delta^{k-1}\mu(W_i) + (k - 1 - \mu(W_i))$$

Thus, for $\delta > 2.4622$, $k = 100000$, we can compute a bound for $\nu_k(V \setminus G)$, which satisfies the JR condition

$$\nu_k(V \setminus G) < \delta^{k-1} = \nu_k(G). \quad \square$$

Lemma 5.5

Since $\tilde{\phi}(k) = \delta \cdot \tilde{\phi}(k-1) + 1$ and $\tilde{\phi}(0) = 1$, $\tilde{\phi}(k) = \frac{\delta^{k+1}-1}{\delta-1}$. Thus $\psi(k) = \frac{\delta^k-1}{\delta-1}$, and we have

$$\frac{\psi(k+m) - \psi(k)}{\psi(m)} = \frac{\delta^{k+m} - \delta^k}{\delta^m - 1} = \delta^k$$

$$\frac{\psi(k+n) - \psi(k)}{\psi(n)} = \frac{\delta^{k+n} - \delta^k}{\delta^n - 1} = \delta^k$$

so they are equal for any m, n, k. $\square$

Lemma 5.6

The proof follows from that of Lemma 5.4. Since Inequality 5.1 is satisfied with equality, the proof of Lemma 5.4 demonstrates that in any period i, performing any substitution of subsets $X \subset W_i$ and $Y \subset V \setminus W_i$, where $\nu_i(X) = \nu_i(Y)$, achieves the same $\nu_k(V \setminus G)$. $\square$

B.6 Section 5.3: Polynomial rules

Theorem 5.3

We start by constructing an example where the rule fails any α-approximation of EJR in the continuous-voter case and then explain how we can convert it to an example with discrete voters.

Pick $\delta_0 = \min(\delta, \min_{k \leq K} \frac{\tilde{\phi}(k)}{\tilde{\phi}(k-1)}) > 1$. Then for all periods i, $\tilde{\phi}(i) \geq \delta_0 \cdot \tilde{\phi}(i-1)$. Let $V = [0, 2n + 1 - \varepsilon]$ for $n > \log_{\delta_0} 2$ and small $\varepsilon > 0$. Let $G = [2n, 2n + 1 - \varepsilon]$ cohesive. The following algorithm defines $V_i \subset V$.

$A := [0, n]$, $B := [n, 2n]$, $S := (A, B)$, $i := 1$, $q := 0$
loop
  $K := \min S[0]$
  while $K < \max S[0]$ **do**
    $V_i := [K, k+1]$; $i \leftarrow i + 1$; $K \leftarrow K + 1$
  $K := \min S[1]$
  while $K < \max S[1]$ **do**
    for $q' \in [2^q]$ **do**
      $V_i := [K + (q'-1)/2^q, K + q'/2^q]$; $i \leftarrow i + 1$
    $K \leftarrow K + 1$
  $V_i := G$; $i \leftarrow i + 1$
  $S \leftarrow (S[1], S[0])$; $q \leftarrow q + 1$

For all i, $v \in V \setminus G$, let $A_i(v) = \{w_i\}$ for $v \in V_i$ and $A_i(v) = \emptyset$ otherwise. We will show that $W_j = V_j$ for all j.

For some i, assume $W_j = V_j$ for $j \in [i-1]$; we show $W_i = V_i$. If $V_i = G$, in period i, all of G approves some candidate, and for all $v \in V \setminus G$, $A_i(v) = \emptyset$, so such candidate wins and $W_i = G$. For the remaining cases, since voters in $V \setminus (G \cup V_i)$ approve nothing, it suffices to show $\nu_i(V_i) > \nu_i(G)$.

Case 1: at the point V_i is defined in the algorithm, $q = 0$. Then for all $v \in G$ and $v' \in V_i$, $\alpha_i(v) = \alpha_i(v') = \tilde{\phi}(i-1)$, and $\mu(V_i) > \mu(G)$, so $\nu_i(V_i) > \nu_i(G)$.

Case 2: $q > 0$ and V_i is defined in the first while loop. At the point V_i is defined in the algorithm, for $v \in G$, consecutive losses are $\text{prop}_i(v) = K - \min S[0]$, while for $v' \in V_i$, $\text{prop}_i(v') \geq K - \min S[0] + 1$, so $\alpha_i(v') > \alpha_i(v)$, and $\mu(V_i) > \mu(G)$, therefore $\nu_i(V_i) > \nu_i(G)$.

Case 3: $q > 0$ and V_i is defined in the second while loop. At the point V_i is defined in the algorithm, for $v \in G$, $\text{prop}_i(v) = n + K - \min S[1]$, while for $v' \in V_i$, $\text{prop}_i(v') \geq n + K - \min S[1] + 2^{q-1}n$. Thus $\alpha_i(v') \geq (\delta_0^n)^{2^{q-1}} \cdot \alpha_i(v) > 2^{2^{q-1}} \cdot \alpha_i(v)$. Since $\mu(V_i) = \frac{1}{2^q}\mu(V_i)$ and $q \geq 1$, $\nu_i(V_i) > 2^{2^{q-1}-q}\nu_i(G) \geq \nu_i(G)$.

G is represented $r - 1$ times in the first $k(r) := r - 1 + \sum_{q=0}^{r-1} n(1 + 2^q)$ periods. For any α, pick $r \in \mathbb{N}$ large enough that $\frac{2^r}{r} > \frac{\alpha}{n} \cdot \frac{2n+1-\varepsilon}{1-\varepsilon}$ and consider the $k = m(r)$ period horizon. Then $\mu(G) > r\alpha \cdot \frac{\mu(V)}{k}$, but any voter in G is only represented at most $r - 1$ times, so the winner sequence fails to be an α-approximation of EJR. For a setup with discrete voters, let $\varepsilon = \frac{1}{2^r}$, then we can divide V into slices with $\mu = \frac{1}{2^r}$ with cohesive preferences in all periods and consider each of these slices to be an individual voter. $\square$

Theorem 5.4

First, note that the linear rule satisfies Inequality 5.1 with exact equality. Since $\tilde{\phi}(k) = k + 1$, $\psi(x) = x$. Then

$$\frac{\psi(k+m) - \psi(k)}{\psi(m)} = \frac{m}{m} = 1 = \frac{n}{n} = \frac{\psi(k+n) - \psi(k)}{\psi(n)}$$

so any Pareto efficient strategy is maximal.

Let G be cohesive and $\mu(G) \geq 2\ell\mu(V)/k$. Observe that in any Pareto efficient strategy, if G remains unsatisfied in period i, for each period i

$$\nu_{i+1}(V \setminus G) = \nu_i(V \setminus G) + \mu(V \setminus G) - \nu_i(G)$$

so therefore

$$\nu_k(V \setminus G) = k \cdot \mu(V \setminus G) - \sum_{i=1}^{k-1} \nu_i(G)$$

Since G is not satisfied in any periods, $\nu_i(G) = i \cdot \mu(G)$, so the sum is $\frac{1}{2}k(k-1)\mu(G)$. In addition, since $\mu(G) \geq 2 \cdot \ell \cdot \mu(V)/k$, $\mu(V \setminus G) \leq \frac{k-2\ell}{2\ell}\mu(G)$, so we have

$$\nu_k(V \setminus G) \leq \frac{1}{2}k(\frac{k-2\ell}{2\ell})\mu(G) - \frac{1}{2}k(k-1)\mu(G) \leq 0$$

which is a contradiction, so G must be satisfied *before* period k.

Lemma 5.7
Since $\tilde{\phi}(x) = (x+1)^p, \psi(x) = x^p$. We can rewrite Inequality 5.1 in the following fashion:

$$\frac{\psi(k+m)-\psi(k)}{m} \cdot \frac{m}{\psi(m)-\psi(0)}$$
$$\geq \frac{\psi(k+n)-\psi(k)}{n} \cdot \frac{n}{\psi(n)-\psi(0)}$$
$$\log\left(\frac{\psi(k+m)-\psi(k)}{m}\right) - \log\left(\frac{\psi(m)-\psi(0)}{m}\right)$$
$$\geq \log\left(\frac{\psi(k+n)-\psi(k)}{n}\right) - \log\left(\frac{\psi(n)-\psi(0)}{n}\right)$$

for any $m < n$, which follows from the fact that $\log \psi'(x)$ is everywhere concave. □

Lemma 5.8
The proof follows similarly from that of Lemma 5.4, except we perform swaps in the opposite direction. In each period i, considering subset $X \subset W_i$ with lower density and subset $Y \subset V \setminus W_i$ with higher density and $\nu_i(X) = \nu_i(Y)$, argue that swapping the preferences of X and Y in period i weakly increases $\nu_k(V \setminus G)$, since Inequality 5.1 holds in the opposite direction. □

Theorem 5.5
Consider the maximal strategy provided by Lemma 5.8. In this maximal strategy, we allocate the non-group voters with the highest density to approve the winners. We wish to bound $\nu_k(V \setminus G)$. Denote $\alpha = 1 + \frac{1}{p}$, and WLOG let $\mu(V) = k/\alpha$. Let G be cohesive with $\mu(G) \geq \alpha\mu(V)/k$.

Suppose a 1-indexed list A_i keeps track of the measures of non-group voters with each possible weight in period i. Then define A_1 to be the list with one entry $k/\alpha - 1$, and each entry j denotes

$$A_i[j] = \mu(\{v \in V \setminus G \mid \alpha_i(v) = \tilde{\phi}(i-j)\})$$

In each period, we can compute the list A_{i+1} from A_i by performing the following algorithm. Add an entry $A_{i+1}[i+1]$ to the end of A_i. Keep track of a budget B, which is initially $\alpha\tilde{\phi}(i-1)$, that represents the remainder of $\nu_i(G)$, and iterate on the list starting at the 1st index until the budget is used up. On each list item j, if $B \geq A[j]\tilde{\phi}(i-j)$ (where the right hand side represents the ν_i value of the non-group voters with density $\tilde{\phi}(i-j)$), then set the list item to zero and add it to the entry $A[i+1]$; continue iterating with the remaining budget. Otherwise, remove $B/\tilde{\phi}(i-j)$ from the list item and add it to the entry $A[i+1]$, then stop iteration.

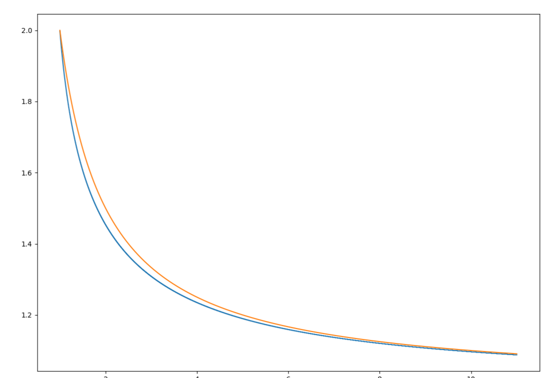
The iteration proceeds from the beginning of the list because under the maximal strategy, the voters with the highest weights are selected to approve the winners, and after they approve, their weights are reset to the lowest weight (which is why their μ is added to the last entry in A). To make this process easier to describe, when some amount of μ is added to the last list entry, say that G has "consumed" these voters.

To show the α-approximation, we must show that $\nu_k(V \setminus G) < \nu_k(G)$, which is the same as the claim that in period k, this iteration will proceed through all of the list entries $A[1], ..., A[k]$.

For a polynomial rule with parameter p, we can approximate the behavior of this discrete-iteration process by a continuous process described by the following differential equation:

$$\frac{dy}{dt} = \left(\frac{t}{t - t(y-k)}\right)^p$$

where y represents the total μ of voters that have been consumed since period 1, and t is a continuous quantity that describes how many periods have passed. Also, we have the initial condition that up to $t = k/\alpha$, $y = t$ (as up to period k, G consumes exactly $\mu = 1$ voters in every period). The function $t(y-k)$ represents the time at which the total μ consumed was $y - k$. This differential equation defines a curve $y = c_k(t)$ such that the curve defined for periods $\beta \cdot k$ satisfies $c_{\beta \cdot k}(t) = \beta c_k(t/\beta)$. The relevant condition to check is whether $\tau := \inf_t\{\frac{dy}{dt} > k/\alpha\} < k$ – if so, G will be able to consume the entire electorate in a single period, and therefore its preferred candidate must win the election. Though this curve does not have a closed-form solution, we can lower and upper bound it by the discrete process that it approximates. In particular, if T is the earliest period such that G consumes the entire electorate in a single period, then $T < \tau < T + 1$. Since the bound $\alpha = 1 + \frac{1}{p}$ is not tight, we can select a large enough number of periods such that $T + 1 < k$, which then shows the theoretical bound for any number of periods (as for any k, $T + 1$ provides an upper bound on τ, and $T < \tau < T + 1 < k$). These bounds allow us to prove the α-approximation for some polynomials; we chose to explore $1.01 < p < 10$, but this technique may also allow us to show the α-approximation for p outside of this interval. The blue line on the below graph shows the JR approximation bound, while the yellow line is the function $1 + 1/x$.



Theorem 5.6
We walk through the proof in more detail. Consider a k-decision instance D and a cohesive group G, $|G| \geq 2\ell \cdot |V|/k$. Since any voter in G may be allowed to be satisfied up to $\ell - 1$ times without the winner sequence satisfying EJR, we allow the adversary to

- Partition G into finitely many subsets G_q;
- On each G_q, select the periods $R_q \subset [k-1]$ in which G_q is satisfied, such that $|R_q| \leq \ell - 1$;
- Choose the preferences of non-group voters $V \setminus G$ such that each G_q is satisfied in exactly periods R_q;

performing this process in a way that maximizes $\nu_k(V \setminus G) - \nu_k(G)$. The third step of this optimization follows the same maximization procedure as the JR adversary; since $\nu_k(G)$ depends only on the subsets G_q and the periods in which they are satisfied (R_q), the adversary must only choose how to allocate voters $V \setminus G$ to the W_i such that $\nu_k(V \setminus G)$ is maximized. Again, as the linear rule satisfies Inequality 5.1 with equality, a version of Lemma 5.6 holds: considering the first two steps of this optimization fixed, every Pareto optimal strategy on the third step is maximal.

For each i, define $H_i = G \setminus W_i$. Since the only criteria of adversarial strategies for EJR is that no voter in G can be satisfied more than $\ell - 1$ times, it is possible that the adversary selects a strategy such that a subset of non-group voters supports the candidate of a subset of G, preventing the preferred candidate that lies in the intersection of voter preferences within G from being selected. For this reason, we allow the adversary to select any arbitrary subset of G to be satisfied in any period. Since H_i is the portion of G left unsatisfied, non-group voters must provide $\nu_i(H_i)$ vote weight to the supported candidate of $G \cap W_i$ so it can win over the preferred candidate of all of G. For every period i, under any Pareto optimal strategy

$$\nu_{i+1}(V \setminus G) = \nu_i(V \setminus G) + \mu(V \setminus G) - \nu_i(H_i)$$

$$\nu_k(V \setminus G) = k \cdot \mu(V \setminus G) - \sum_{i=1}^{k-1} \nu_i(H_i)$$

Also, observe that

$$\nu_k(G) + \sum_{i=1}^{k-1} \nu_i(G \setminus H_i) = k \cdot \mu(G)$$

by the same logic as the weight for $V \setminus G$: under the linear rule, in each period i, the weight of voters $v \in G$ who are satisfied is removed from $\nu_{i+1}(G)$, and then $\mu(G)$ weight is added to G. Since every voter in G is represented at most $\ell - 1$ times

$$\sum_{i=1}^{k-1} \nu_i(G) \geq \ell \cdot \frac{1}{2} \cdot \frac{k}{\ell} \cdot (\frac{k}{\ell} + 1)\mu(G) \geq \frac{k^2}{2\ell}\mu(G)$$

where the bound is attained when the periods in which voters in G are satisfied evenly divide the k total periods.

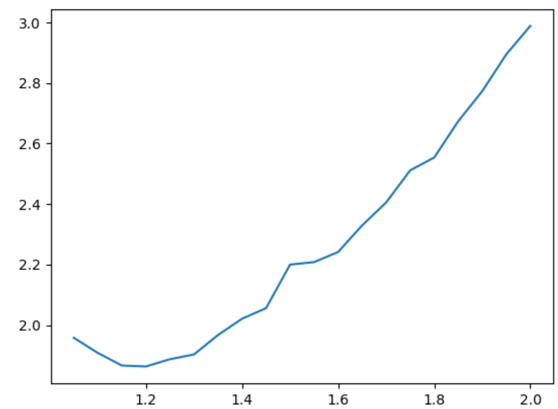
$$\begin{aligned} &\nu_k(V \setminus G) - \nu_k(G) \\ &= k \cdot \mu(V \setminus G) - k \cdot \mu(G) - \sum_{i=1}^{k-1} \nu_i(G) \\ &\leq k(\frac{k}{2\ell})\mu(G) - k \cdot \mu(G) - \frac{k^2}{2\ell}\mu(G) < 0 \end{aligned}$$

so G must be represented in period k.

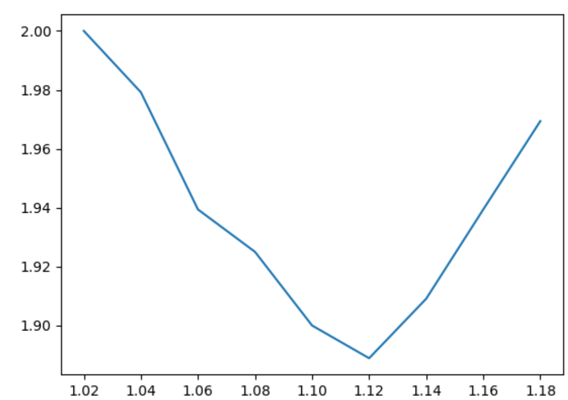
C Polynomial rules and EJR approximations

While polynomial rules for $p > 1$ achieve a better approximation of JR than the linear rule with $p = 1$, larger polynomials do not achieve a better EJR approximation than the linear rule for the same reason that discount rules fail on this metric. Increased growth in vote weights after several consecutive losses means that groups that may deserve to be represented relatively frequently will be de-prioritized. One additional difficulty of studying polynomial rules is that it is hard to reason about the optimal adversarial strategy because the strategy space (as described in the proof of Theorem 5.6) is very large. Unlike the linear case, the total vote weights of groups are not easily computable by adding the voter measure and subtracting an offset for the subset of a group that is satisfied.

For $k \leq 100$ periods, we plot below the worst-case EJR approximation (on the y-axis) against the polynomial parameter p (on the x-axis) where the adversary is restricted to using the strategy that optimizes against the JR approximation and is optimal for the EJR approximation in the linear case, in which the adversary attempts to prolong each additional group representation by as many periods as possible.



For $k \leq 100$ periods, we plot below the worst-case EJR approximation (on the y-axis) against the polynomial parameter p (on the x-axis) where the adversary is restricted to using strategies in which for the spacing between the first $k-1$ times G is represented is fixed.



These graphs offer some evidence that polynomial rules for $p > 1$ may be able to achieve better approximations for EJR than the linear rule for $p = 1$. More sophisticated adversarial strategies may prove to worsen the approximations achievable via polynomial rules. Notably, however, these plots line up with the superior performance of the polynomial rule with

$p = 1.1$ in our simulations.

D Simulations

D.1 Methodology Details

We provide further details discussing our perpetual voting simulations, building on the presentation in Section 6. Our intention behind constructing these simulations was to mimic realistic perpetual voting settings while also effectively demonstrating EJR satisfaction.

In generating initial voter preferences for decision sequences with N voters and M candidates, we specifically set each voter to approve $\mathcal{N}(M/4, M/10)$ of the candidates, where $\mathcal{N}$ denotes the Normal distribution. This reflects a realistic assumption that voters should approve similar numbers of candidates. We also randomly set candidates to be chosen with different likelihoods: We generate for each candidate a random number uniformly in $[0, 1]$ to create an array of relative likelihoods, normalize the array to sum to 1, and then generate each voter's preferences via a random choice with this array used to denote choice probabilities (via `numpy.random.choice`).

This method operates similarly to the *p-Impartial Culture* method described in Bredereck et al [2019], but further specifying each candidate's likelihoods of being chosen and the average number of candidates chosen. While this method differs from those used in the simulations by Lackner [2020] and Bredereck et al [2019], setting each voter to approve approximately a quarter of the candidates aligns with Lackner [2020], in which an average of 1.8 out of 5 (36%) candidates were approved every period, and Bredereck et al [2019], in which an average of between 0% and 50% of candidates were approved every period. We also believe that this method's inclusion of factors such as candidates' relative approval likelihoods, which are arguably analogous to popularity levels, makes it more realistic than specifying preferences through random subset generation, as is done in Bulteau et al [2021].

Our method of changing preferences between periods using the Geometric distribution is also comparable to Bulteau et al [2021], which changes votes for currently-approved or disapproved candidates independently; however, using a distribution allows us to succinctly summarize how many preferences change. Note that by the Geometric distribution, each voter changes their vote for 1 candidate on average between periods. We also clip the distribution's values to $[0, M]$ since the number of votes changed must lie in that range. Broadly, it may be of further interest to examine whether or not these simulation results change when different aspects of this preference generation process are modified.

In explicitly constructing cohesive groups to test EJR, our choice to modify the decision sequence to change the group's preferences as described (members in the group approve only one candidate in each period, which other voters do not approve) is intentional and done to best test the performance of EJR in edge cases that arguably remain realistic. Although this construction leaves the group members with identical preferences in each period that are isolated from the rest of the voters, this setting is feasibly comparable to practical situations, such as by interpreting the group as a minority bloc with alternative views. To satisfy EJR, the single candidate approved by the group would have to win sufficiently many times, despite being approved by only the group. Rules whose formulations do not naturally lead to the identification of this group as deserving representation, such as plain approval voting or GreedyCC, can thus empirically be shown to fail to satisfy EJR and not demonstrate proportional representation of the group, as seen in Figure 1.

These differences between rules are noticably less pronounced when we relax constraints on the preferences of the constructed group. Intuitively, in each period, if the group can also approve other candidates, the likelihood that any voter in the group is satisfied in each period increases since they now approve strictly more candidates. If voters outside the group can also approve the candidate unanimously approved by the group, the likelihood that the candidate is elected increases as it receives strictly more support from voters. In both cases, the overall satisfaction of the group increases, causing general voting rules—even plain approval voting—to be much more successful in satisfying EJR. We demonstrate this below.

Because these experiments only consider one group and its fulfillment of EJR in each decision sequence, this allows us to consider many voters ($N = 1000$). The parameter ranges for values of M and k used are similar to those used by Lackner [2020] and Bulteau et al [2021].

D.2 Additional Simulations

Using the same methodology for creating a cohesive group and testing for its EJR satisfaction under various rules as in Section 6, we considered EJR satisfaction results for additional settings of N, M, k, and ℓ outside of those given in Table 1. We considered using larger values of the number of candidates M, including $M \in \{10, 20\}$; modifying the average number of candidates approved by each voter in initial preference generation to values like $M/5$ or $M/3$ of the candidates; and increasing k and ℓ to other reasonable values, including $k \in \{30, 40, 50\}$ and $\ell \in \{5, 10, 15\}$.

Running simulations with combinations of these additional parameter values led to similar results and trends in rules' EJR satisfaction. AV continued to fail to satisfy EJR in the vast majority of instances, while Perpetual Phragmen and Equal Shares satisfied EJR in all instances. We found overall that the trends exhibited by GreedyCC, Perpetual PAV, discount, and polynomial rules depended on the ratio ℓ/k; this is a generalization of the trend exhibited in Table 1, which was based increasing values of ℓ as fixed values of k were considered. GreedyCC satisfied JR (equivalent to the cases with $\ell = 1$), but its performance dropped with increasing ℓ/k. Perpetual PAV satisfied EJR increasingly with increasing ℓ/k. Discount and polynomial rules exhibited decreasing performance with increasing ℓ/k, with lower values of δ in discount rules and lower values of n in polynomial rules performing better in comparison. Given the close similarity of these additional results for these larger ranges of parameters tested, we do not provide additional summary tables.

Relaxing some of the restrictions on the preferences of the constructed cohesive group being studied led to substantial increases in rules' satisfaction of EJR. Indeed, if either of the restrictions that

(a) each voter in the group approves only one shared candidate, and no other candidates, in each period;

(b) voters not in the group cannot approve the candidate approved by the group in each period;

were relaxed, then all rules except for GreedyCC satisfied EJR in all instances. Thus adding these additional restrictions on the constructed group were necessary to empirically differentiate the rules.

In this vein, we additionally followed the same process used to generate Figure 1 with the above two conditions for the cohesive group relaxed in simulations, plotting maximum group voter satisfaction against group size as a proportion of N for $M = 10$, $N = 100$, $k = 50$ and group size in $[2, 40]$ for 100 iterations each, producing Figures 2-4. As can be seen, if the group's voting preferences are not as isolated by relaxing restrictions, then in the vast majority of cases all rules better satisfy the group for all group sizes.

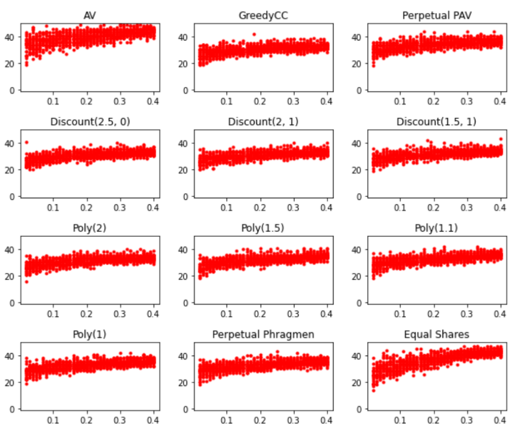


Figure 2: **Maximum Group Voter Satisfaction (vertical axis) versus Group Size (horizontal axis), with above condition (a) removed.** Allowing voters in the constructed group to vote for other candidates in addition to the shared candidate leads to high satisfaction levels for all rules and group sizes.

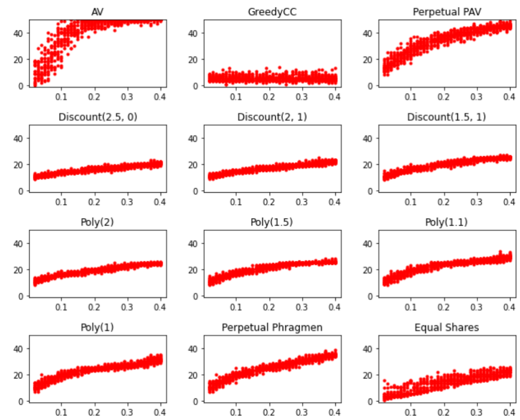


Figure 3: **Maximum Group Voter Satisfaction (vertical axis) versus Group Size (horizontal axis), with above condition (b) removed.** Allowing voters outside the constructed group to vote for the group's preferred candidate also leads to higher maximum satisfactions, although AV and GreedyCC do not demonstrate the desired linear trend between maximum satisfaction and group size. Interestingly, this is a situation where GreedyCC does worse on average.

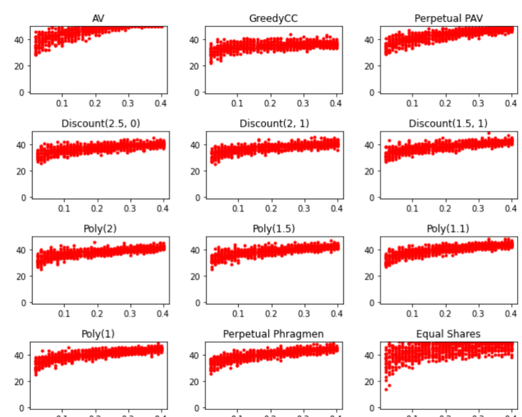


Figure 4: **Maximum Group Voter Satisfaction (vertical axis) versus Group Size (horizontal axis), with above conditions (a) and (b) removed.** Relaxing both restrictions also leads to high satisfaction levels across the board.